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Possible Species of Ferromagnetic, Ferroelectric, and Ferroelastic Crystals

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A ferromagnetic, ferroelectric, or ferroelastic crystal is called full or partial, according to whether all or not all but some of its orientation states are different in spontaneous magnetization vector, spontaneous polarization vector, or spontaneous strain tensor. In previous theories – for nonmagnetic crystals – the concept of "species" was introduced, a determination was made of all possible species of full ferroelectrics and of full ferroelastics, and those species were found in which ferroelectricity and ferroelasticity coexist and completely couple with each other. These theories are now extended to cover magnetic crystals in addition to nonmagnetic crystals and to cover the partial in addition to the full. A determination is made of all possible species of full ferromagnetics, partial ferromagnetics, full ferroelectrics, partial ferroelectrics, full ferroelastics, and partial ferroelastics, and it is found out in which of these species two or all of ferromagnetism, ferroelectricity, and ferroelasticity should couple completely or incompletely with each other.

1. INTRODUCTION

A crystal is provisionally referred to as being "ferroic" when it has two or more orientation states in the absence of magnetic field, electric field, and mechanical stress and can shift from one to another of these states by means of a magnetic field, an electric field, a mechanical stress, or a combination of these. Here any two of the orientation states are identical or enantiomorphous in structure but are different with respect to direction of arrangement of atoms which may possess an electric charge, an electric dipole moment, and a magnetic dipole moment.

The ferromagnetic crystals are those ferroic crystals whose orientation states are all different in spontaneous magnetization vector. The ferroelectric crystals are those ferroic crystals whose orientation states are all different in spontaneous polarization vector. There are ferroic crystals whose orientation states are all different in spontaneous strain tensor. (In this paper, "spontaneous strain tensor" means simply "strain tensor at null stress.") They are called ferroelastic.[1] In the ferromagnetic crystals the shift between any two orientation states can be brought about by a magnetic field; in the ferroelectric crystals the shift between any two orientation states can be brought about by an electric field, and in the ferroelastic crystals, by a mechanical stress.

Precisely speaking, the above-defined ferromagnetics, ferroelectrics, and ferroelastics are *full* ferromagnetics, full ferroelectrics, and full ferroelastics, respectively. Ferroics are possible, not all but some of whose orientation states are different in spontaneous magnetization vector, spontaneous polarization vector, or spontaneous strain tensor. We call these ferroics *partial* ferromagnetics, partial ferroelectrics, or partial ferroelastics, and regard them as a kind of ferromagnetics, ferroelectrics, or ferroelastics. (For magnetic crystals, fullness and partiality of ferroelectricity and of ferroelasticity are defined a little otherwise. See Sec. 2.)

In previous papers[1-3] for nonmagnetic crystals, we introduced the concept of "species" and found that there are, in all, 88 possible species of full ferroelectrics and 94 possible species of full ferroelastics, of which 42 species are common to both, i.e., simultaneously full ferroelectric and full ferroelastic. In this paper we undertake to extend the previous papers to cover magnetic crystals in addition to nonmagnetic crystals and to cover the partial in addition to the full ones. For all crystals, magnetic and nonmagnetic, we will determine all possible species oi full ferromagnetics, partial ferromagnetics, full ferroelectrics, partial ferroelectrics, full ferroelastics, and partial ferroelastics, and will find out which of these species are common, i.e., have two or all of (full or partial) ferromagnetism, ferroelectrictiy, and ferroelasticity simultaneously.

2. FUNDAMENTAL CONCEPTS

We denote the rotation group pertaining to space (including space inversion) by Γ_r and the time-inversion group that consists of the identity and time inversion[4] by $1'$. The direct product of Γ_r and $1'$ is usually written $\Gamma_r \times 1'$.

Let S and S' be two arbitrary orientation states of a ferroic crystal. They are identical or enantiomorphous in structure. This means mathematically that S' is to be obtained by performing a certain operation of $\Gamma_r \times 1'$ upon S. We refer to this operation as an "F operation from S to S'" of the ferroic crystal. (F is the initial of ferroic.) In particular, when S and S' are identical, an F operation from S to S' or S itself is just an element of the point group ($\subset \Gamma_r \times 1'$ and not necessarily $\subset \Gamma_r$) of S. In general, there is more than one F operation from S to S'.

Consider, for instance, the room-temperature phase of $BaTiO_3$ known well as a full ferroelectric. This has, in all, six orientation states in which the spontaneous polarization vectors are antiparallel or perpendicular to one another. We set a system of rectangular coordinate axes x, y, z and designate the orientation states with spontaneous polarization vector in the $+x$, $-x$, $+y$, $-y$, $+z$, and $-z$ directions as S_1, S_2, S_3, S_4, S_5, and S_6, respectively. Then, as examples of F operations from S_1 to S_2, we have space inversion and the space reflection across the yz plane. As examples of F operations from S_1 to S_3, we have the 180° space rotation about the [110] axis and the 120° space rotation about the [111] axis. Also, consider the room-temperature phase of iron known well as a full ferromagnetic. This has, in all, six orientation states the spontaneous magnetization vectors in which are antiparallel or perpendicular to one another. We set a system of rectangular coordinate axes x, y, z, and designate the orientation states with spontaneous magnetization vector in the $+x$, $-x$, $+y$, $-y$, $+z$, and $-z$ directions as S_1, S_2, S_3, S_4, S_5, and S_6, respectively. Then, as examples of F operations from S_1 to S_2, we have time inversion and the 180° space rotation about the y axis. As examples of F operations from S_1 to S_3, we have the 90° space rotation about the z axis and the 120° space rotation about the [111] axis.

Let a ferroic crystal have q orientation states in all and let S be one of them. We refer to a set of q F operations each from S to each orientation state as a "set of representative F operations on S" of the ferroic crystal. In general, there is more than one set of representative F operations on S.

For a crystal to become ferroic, it is not sufficient that some orientation states can be assigned to the crystal. Any two of these states, in addition, must be able to change to each other through only slight movements of the atomic nuclei. So, in general, a ferroic crystal may be regarded as a slight modification of a certain nonferroic ideal crystal which is referred to as the "prototype" of that ferroic crystal: (a) All the F operations from all to all of the orientation states of the ferroic crystal are to be contained in the point group of the prototype; (b) any element of the point group of the prototype is to be an F operation for the ferroic crystal or, more exactly speaking, when f is an element of the point group of the prototype and S is an orientation state of the ferroic crystal, the result from operation of f on S is always to be as possible an orientation state of the ferroic crystal as S. (This second item, on the one hand, prevents the prototype's having superfluous symmetry elements not connected with the phenomenon of ferroicity and, on the other hand, defines a complete set of orientation states.)

For instance, the tetragonal, orthorhombic, and rhombohedral phases of $BaTiO_3$, which are all full ferroelectric, can be imagined to be derivatives of a common prototype whose unit cell is cubic and comprises barium atoms at its corners, oxygen atoms exactly at its face centers, and a titanium atom exactly at its body center.

On varying temperature, a ferroic crystal may usually make a phase transformation to a nonferroic phase having the same symmetry as the prototype; we refer to this phase as the "prototypic phase" of that ferroic crystal (or, sometimes, as the paramagnetic, paraelectric, or paraelastic phase of that ferroic crystal, when the ferroic crystal is ferromagnetic, ferroelectric, or ferroelastic, respectively). For instance, the cubic phase of $BaTiO_3$ is prototypic. $LiH_3(SeO_3)_2$ is a ferroelectric crystal without a prototypic phase (under an ordinary atmospheric pressure).[5] In KNO_3, three phases are known: What is called phase III is trigonal and ferroelectric; phase I is trigonal and nonferroelectric; phase II is orthorhombic and nonferroelectric.[6] Phase I is the prototypic of phase III, and phase II is not. In iron, on raising temperature, the ferromagnetic tetragonal phase transforms to a nonferromagnetic bcc phase at 770 °C and then to a nonferromagnetic fcc phase at 910 °C. The bcc phase is the prototypic of the ferromagnetic phase, and the fcc phase is not.

A point group including time inversion as an element and a crystal belonging to such a point group are referred to as being "time symmetric." Consider a non-time-symmetric ferroic crystal. Let S be an orientation state of this ferroic crystal, and denote the result from performance of time inversion upon S by S'. S' must be different from S. It is obvious that the spatial configuration of the nuclei in S' is identical with that in S. Therefore the shift from S to S' is considered not to be energetically difficult. Thus S' should be as possible an orientation state of the ferroic crystal as S, and so the prototype of this ferroic crystal should be time symmetric. On the other hand, the prototype of every time-symmetric ferroic crystal is, of course, time symmetric. After all, it turns out that the prototype of every ferroic crystal, whether time symmetric or not, is time symmetric.

Magnetic crystals (or, more precisely speaking, magnetically ordered crystals) are all lacking in time inversion as an element of their space group ($\subset \Gamma_r \times \Gamma_t \times 1'$ where Γ_t is the translation group pertaining to space), while all nonmagnetic crystals have time inversion as an element of their space group.[4] Hence all nonmagnetic crystals are time symmetric, but not all magnetic crystals are non-time symmetric, in other words, some magnetic crystals may be time symmetric. (It is possible that a magnetic crystal, whereas lacking in time inversion as an element of its space group, possesses time inversion as an element of its point group.) It is considered that the prototype of every ferroic crystal is not merely time symmetric but nonmagnetic.

There are, in all, 122 point groups $\subset \Gamma_r \times 1'$.[4] Ninety of them are not time symmetric. The remaining 32 are time symmetric and equal to the direct products of $1'$ and the well-known 32 point groups $\subset \Gamma_r$. Every time-symmetric point group $\subset \Gamma_r \times 1'$ will be symbolized by tailing the symbol for the corresponding point group $\subset \Gamma_r$ with the symbol $1'$. $\bar{1}1'$ and $m3m1'$ are examples. $11'$ is not used because it means the same as $1'$. In the following, $\bar{1}$, $m3m$, etc., will represent non-time-symmetric point groups and so they must be distinguished from $\bar{1}1'$, $m3m1'$, etc.

The time-symmetric and the non-time-symmetric point groups are also often called nonmagnetic and magnetic point groups, respectively.[4] However, it should be noted that not all magnetic crystals belong to non-time-symmetric point groups.

In a non-time-symmetric ferroic crystal, two orientation states between which time inversion is an F operation are referred to as being "time conjugate" to each other. The orientation states of every non-time-symmetric ferroic crystal can be divided into pairs of time-conjugate orientation states. Since, as is well known, spontaneous polarization vector and spontaneous strain tensor are invariant under time inversion, any two time-conjugate orientation states must be equal in these quantities to each other. Therefore any non-time-symmetric ferroic crystal cannot be *full* ferroelectric or *full* ferroelastic in the sense that *all its orientation states* are different in spontaneous polarization vector or spontaneous strain tensor. A non-time-symmetric ferroic crystal will be said to be full ferroelectric or full ferroelastic, when any two non-time-conjugate orientation states of it are different in spontaneous polarization vector or spontaneous strain tensor. For the time-symmetric ferroic crystals, the definition of full ferroelectricity and of full ferroelasticity are the same as given in Sec. 1. The definition of full ferromagnetism is unaltered. Since, as is well known, spontaneous magnetization vector is reversed by time inversion, every ferromagnetic crystal, whether full or partial, must be non-time symmetric, and any time-conjugate orientation states of it must be antiparallel with respect to spontaneous magnetization vector.

Ferroic crystals are referred to as belonging to the same "species" when they are the same with respect to (i) their own point group (ferroic point group), (ii) the point group of their prototype (prototypic point group), and (iii) the correspondence between the elements of the ferroic point group and the elements of the prototypic point group. (The meaning of the third item will shortly become clear as we proceed.) We represent each species by a compound symbol that comprises the letter F (meaning ferroic) at its center, the symbol for the prototypic point group on the left of F, and the symbol for the ferroic point group on the right of F.

In many cases the correspondence between the elements of the prototypic point group and the elements of the ferroic point group is unique, so that the correspondence need not be indicated explicitly in the species symbol. The species $\bar{4}2m1'\text{F}\,m'm'2$ is such a case; here the m' planes and the diad axis of the ferroic point group should correspond to (or originate from) the m' planes and the $\bar{4}$ axis of the prototypic point group, respectively. In some cases, however, we need to indicate difference in correspondence. The species $\bar{4}2m1'\text{F}2'(p)$ and $\bar{4}2m1'\text{F}2'(s)$ are such cases. In the former species the $2'$ axis of the ferroic point group corresponds to the $2'$ axis parallel to the tetragonal unique axis of the prototypic point group. In the latter species the $2'$ axis of the ferroic point group corresponds to one of the $2'$ axes perpendicular to the tetragonal unique axis of the prototypic point group. (See Sec. 4 as to why the letters p and s are used.)

3. THEOREM ON NUMBER OF ORIENTATION STATES

When S is an orientation state of a ferroic crystal and f is an operation of $\Gamma_r\times1'$, we express the result from performance of f upon S as fS. When an orientation state S is identical with an orientation state S', we write $S=S'$. We have the following theorems.

Theorem 1. When H is the point group of an orientation state S and f is an F operation from S to another orientation state S', the set of all F operations from S to S' is equal to fH.

Proof. It holds that $fS=S'$. Let g be an arbitrary F operation from S to S'; then $gS=S'$. Using these equations, we have

$$(f^{-1}g)S=f^{-1}(gS)=f^{-1}(fS)=(f^{-1}f)S=1S=S.$$

(f^{-1} is the inverse of f. 1 in $1S$ is the identity.) Thus since $f^{-1}g$ keeps S unaltered, it must be equal to a certain element of H: $f^{-1}g=h\in H$. Therefore, $g=fh\in fH$. Conversely, when h is an arbitrary element of H, we have

$$(fh)S=f(hS)=fS=S',$$

so that fh is an F operation from S to S'. (QED)

Theorem 2. When H is the point group of an orientation state S and f is an F operation from S to another orientation state S', the point group of S' is equal to fHf^{-1}.

Proof. It holds that $fS=S'$. Let h' be an arbitrary element of the point group of S'; then $h'S'=S'$. Using these equations, we have

$$f^{-1}h'fS=f^{-1}h'S'=f^{-1}S'=f^{-1}fS=S,$$

so that $f^{-1}h'f$ must be equal to a certain element of H: $f^{-1}h'f=h\in H$. Therefore, $h'=fhf^{-1}\in fHf^{-1}$. Conversely, when h is an arbitrary element of H, we have

$$fhf^{-1}S'=fhS=fS=S',$$

so that fhf^{-1} is an element of the point group of S'. (QED)

In group algebra, fHf^{-1} is said to be conjugate to H. The point groups of all the orientation states of a ferroic crystal, thus, are conjugate to one another. When we say "the point group of the ferroic crystal," it does not matter which this point group is of these and other conjugate point groups.

Theorem 3. The number of orientation states equals the order of the prototypic point group divided by the order of the ferroic point group.

Proof. Let G be the point group of the prototype, q be the number of orientation states, H be the point group of an orientation state S, and $\{f_1, f_2, \ldots, f_q\}$ be a set of representative F operations on S. According to Theorem 1, the set of all F operations from S to f_iS is equal to f_iH. The number of elements included in f_iH is, obviously, equal to the number of elements included in H, i.e., the order of H. When $i\neq j$, f_iH and f_jH cannot have any common element. Since the union of $f_1H, f_2H, \ldots, f_qH$ must be contained in G and since, conversely, any element of G must belong to the union of $f_1H, f_2H, \ldots, f_qH$, (owing to the second item in the definition of prototype), the union of $f_1H, f_2H, \ldots, f_qH$ must be the same as G. After all, it is evident that q is equal to the order of G divided by the order of H. (QED)

We can write symbolically

$$G=f_1H+f_2H+\cdots+f_qH.$$

In group algebra, this relation is called the resolution of G into left cosets with respect to H.

Since the orders of all the point groups are known, Theorem 3 enables us to evaluate directly the number of orientation states in any species of ferroic crystals. Ferroic crystals belonging to the same species have the same number of orientation states.

4. POSSIBLE SPECIES

We assume the following: (a) Every time-symmetric point group can become the prototypic point group in some species of ferroic crystals. (b) When a prototypic point group is specified, every proper subgroup of it can become the ferroic point group in some species with this prototypic point group. (c) When a prototypic and a ferroic point group are specified, all different ways in which the elements of the ferroic point group correspond to the elements of the prototypic point group give so many possible species.

On the basis of these assumptions, we are able to determine all possible species of ferroic crystals. The results are shown in the second column of Table I. All the possible species are 773 in number. The third column gives the number of orientation states in each species. When the ferroic point group is not time symmetric, the number of orientation states is expressed, for convenience sake, in the form of the number of pairs of time-conjugate orientation states multiplied by 2.

In the second column, p and s in species symbols are the initials of "principal" and "side," respectively. When the prototype belongs to a noncubic system, p means that the crystallographic unique axis or an important axis of the ferroic itself is along the crystallographic unique axis of the prototype, and s means that the crystallographic unique axis or an important axis of the ferroic itself is perpendicular to the crystallographic unique axis of the prototype. When the prototype belongs to the cubic system, p means that the unique axis or an important axis of the ferroic is along one of the cubic principal axes of the prototype, and s means that the unique axis or an important axis of the ferroic is along a face diagonal of the cubic lattice of the prototype.

Comments below are made on some species that contain p and/or s in their symbols. FG and PG are abbreviations of "the ferroic point group" and "the prototypic point group," respectively.

In No. 116, the diad axis of FG is along the tetrad axis of PG; No. 117, the diad axis of FG is along a diad axis perpendicular to the tetrad axis of PG; Nos. 118, 120, similar to No. 116; Nos. 119, 121, similar to No. 117; No. 123, the pure diad axis of FG is along the tetrad axis of PG; No. 124, the pure diad axis of FG is along a diad axis perpendicular to the tetrad axis of PG.

In No. 152, the diad axis of FG is along the $\bar{4}$ axis of PG; No. 153, the diad axis of FG is along a diad axis perpendicular to the $\bar{4}$ axis of PG; Nos. 154, 156, similar to No. 152; Nos. 155, 157, similar to No. 153; No. 162, the pure diad axis of FG is along the $\bar{4}$ axis of PG; No. 163, the pure diad axis of FG is along a diad axis perpendicular to the $\bar{4}$ axis of PG.

In No. 181, the diad axis of FG is along the tetrad axis of PG; No. 182, the diad axis of FG is along a diad axis perpendicular to the tetrad axis of PG; Nos. 183, 185, similar to No. 181; Nos. 184, 186, similar to No. 182; No. 187, the mirror plane of FG is along the mirror plane perpendicular to the tetrad axis of PG; No. 188, the mirror plane of FG is along a mirror plane parallel to the tetrad axis of PG; Nos. 189, 191, similar to No. 187; Nos. 190, 192, similar to No. 188; No. 193, the diad axis of FG is along the tetrad axis of PG or, in other words, the mirror plane of FG is along the mirror plane perpendicular to the tetrad axis of PG; No. 194, the diad axis of FG is along a diad axis perpendicular to the tetrad axis of PG or, in other words, the mirror plane of FG is along a mirror plane parallel to the tetrad axis of PG; Nos. 195, 197, 199, 201, similar to No. 193; Nos. 196, 198, 200, 202, similar to No. 194; No. 204, the pure diad axis of FG is along the tetrad axis of PG; No. 205, the pure diad axis of FG is along a diad axis perpendicular to the tetrad axis of PG; No. 207, the diad axis of FG is along the tetrad axis of PG; No. 208, the diad axis of FG is along a diad axis perpendicular to the tetrad axis of PG; No. 209, the m' plane of FG is along a mirror plane parallel to the tetrad axis of PG, and the pure mirror plane of FG is also along a mirror plane parallel to the tetrad axis of PG (the $2'$ axis of FG is along the tetrad axis of PG); No 210, the m' plane of FG is along the mirror plane perpendicular to the tetrad axis of PG, the pure mirror plane of FG is along a mirror plane parallel to the tetrad axis of PG (the $2'$ axis of FG is along a diad axis perpendicular to the tetrad axis of PG); No. 211, the m' plane of FG is along a mirror plane parallel to the tetrad axis of PG, and the pure mirror plane of FG is along the mirror plane perpendicular to the tetrad axis of PG (the $2'$ axis of FG is along a diad axis perpendicular to the tetrad axis of PG); Nos. 212, 214, similar to No. 207; Nos. 213, 215, similar to No. 208; No. 217, the m' plane of FG is along the mirror plane perpendicular to the tetrad axis of PG; No. 218, the m' plane of FG is along a mirror plane parallel to the tetrad axis of PG; No. 219, the pure mirror plane of FG is along the mirror plane perpendicular to the tetrad axis of PG; No. 220, the pure mirror plane of FG is along a mirror plane parallel to the tetrad axis of PG.

In No. 365, the diad axis of FG is along the hexad axis of PG; No. 366, the diad axis of FG is along a diad axis perpendicular to the hexad axis of PG; Nos. 367, 369, similar to No. 365; Nos. 368, 370, similar to No. 366; No. 372, the pure diad axis of FG is along the hexad axis of PG; No. 373, the pure diad axis of FG is along a diad axis perpendicular to the hexad axis of PG.

In No. 414, the mirror plane of FG is along the mirror plane perpendicular to the $\bar{6}$ axis of PG; No. 415, the mirror plane of FG is along a mirror plane parallel to the $\bar{6}$ axis of PG; Nos. 416, 418, similar to No. 414; Nos. 417, 419, similar to No. 415; No. 421, the m' plane of FG is along the mirror plane perpendicular to the $\bar{6}$ axis of PG, and the pure mirror plane of FG is along a mirror plane parallel to the $\bar{6}$ axis of PG; No. 422, the m' plane of FG is along a mirror plane parallel to the $\bar{6}$ axis of PG, and the pure mirror plane of FG is along the mirror plane perpendicular to the $\bar{6}$ axis of PG.

TABLE I. Possible species of ferroic crystals, their number of orientation states, and their relationships to ferromagnetism, ferroelectricity, and ferroelasticity.

No.	Species	Number of states	Ferromagnetic	Ferroelectric	Ferroelastic
1	$1'\mathrm{F}1$	1×2	Full	···	···
2	$\bar{1}1'\mathrm{F}1$	2×2	P	Full	···
3	$\bar{1}1'\mathrm{F}1'$	2	···	Full	···
4	$\bar{1}1'\mathrm{F}\bar{1}$	1×2	Full	···	···
5	$\bar{1}1'\mathrm{F}\bar{1}'$	1×2	···	···	···
6	$21'\mathrm{F}1$	2×2	Full	Full	Full
7	$21'\mathrm{F}1'$	2	···	Full	Full
8	$21'\mathrm{F}2$	1×2	Full	···	···
9	$21'\mathrm{F}2'$	1×2	Full	···	···
10	$m1'\mathrm{F}1$	2×2	Full	Full	Full
11	$m1'\mathrm{F}1'$	2	···	Full	Full
12	$m1'\mathrm{F}m$	1×2	Full	···	···
13	$m1'\mathrm{F}m'$	1×2	Full	···	
14	$2/m1'\mathrm{F}1$	4×2	P	Full	P
15	$2/m1'\mathrm{F}1'$	4	···	Full	P
16	$2/m1'\mathrm{F}\bar{1}$	2×2	Full	···	Full
17	$2/m1'\mathrm{F}\bar{1}'$	2×2	···	···	Full
18	$2/m1'\mathrm{F}\bar{1}1'$	2	···	···	Full
19	$2/m1'\mathrm{F}2$	2×2	P	Full	···
20	$2/m1'\mathrm{F}2'$	2×2	P	Full	···
21	$2/m1'\mathrm{F}21'$	2	···	Full	···
22	$2/m1'\mathrm{F}m$	2×2	P	Full	···
23	$2/m1'\mathrm{F}m'$	2×2	P	Full	···
24	$2/m1'\mathrm{F}m1'$	2	···	Full	···
25	$2/m1'\mathrm{F}2/m$	1×2	Full	···	···
26	$2/m1'\mathrm{F}2'/m$	1×2	···	···	···
27	$2/m1'\mathrm{F}2/m'$	1×2	···	···	···
28	$2/m1'\mathrm{F}2'/m'$	1×2	Full	···	···
29	$2221'\mathrm{F}1$	4×2	Full	Full	Full
30	$2221'\mathrm{F}1'$	4	···	Full	Full
31	$2221'\mathrm{F}2$	2×2	P	Full	Full
32	$2221'\mathrm{F}2'$	2×2	Full	Full	Full
33	$2221'\mathrm{F}21'$	2	···	Full	Full
34	$2221'\mathrm{F}222$	1×2	···	···	···
35	$2221'\mathrm{F}2'2'2$	1×2	Full	···	···
36	$mm21'\mathrm{F}1$	4×2	Full	Full	Full
37	$mm21'\mathrm{F}1'$	4	···	Full	Full
38	$mm21'\mathrm{F}2$	2×2	P	···	Full
39	$mm21'\mathrm{F}2'$	2×2	Full	···	Full
40	$mm21'\mathrm{F}21'$	2	···	···	Full
41	$mm21'\mathrm{F}m$	2×2	P	Full	Full
42	$mm21'\mathrm{F}m'$	2×2	Full	Full	Full
43	$mm21'\mathrm{F}m1'$	2	···	Full	Full
44	$mm21'\mathrm{F}mm2$	1×2	···	···	···
45	$mm21'\mathrm{F}m'm2'$	1×2	Full	···	···
46	$mm21'\mathrm{F}m'm'2$	1×2	Full	···	···
47	$mmm1'\mathrm{F}1$	8×2	P	Full	P
48	$mmm1'\mathrm{F}1'$	8	···	Full	P
49	$mmm1'\mathrm{F}\bar{1}$	4×2	Full	···	Full
50	$mmm1'\mathrm{F}\bar{1}'$	4×2	···	···	Full
51	$mmm1'\mathrm{F}\bar{1}1'$	4	···	···	Full
52	$mmm1'\mathrm{F}2$	4×2	P	P	P
53	$mmm1'\mathrm{F}2'$	4×2	P	P	P
54	$mmm1'\mathrm{F}21'$	4	···	P	P
55	$mmm1'\mathrm{F}m$	4×2	P	Full	P
56	$mmm1'\mathrm{F}m'$	4×2	P	Full	P
57	$mmm1'\mathrm{F}m1'$	4	···	Full	P
58	$mmm1'\mathrm{F}2/m$	2×2	P	···	Full
59	$mmm1'\mathrm{F}2'/m$	2×2	···	···	Full
60	$mmm1'\mathrm{F}2/m'$	2×2	···	···	Full
61	$mmm1'\mathrm{F}2'/m'$	2×2	Full	···	Full
62	$mmm1'\mathrm{F}2/m1'$	2	···	···	Full
63	$mmm1'\mathrm{F}222$	2×2	···	···	···
64	$mmm1'\mathrm{F}2'2'2$	2×2	P	···	···
65	$mmm1'\mathrm{F}2221'$	2	···	···	···
66	$mmm1'\mathrm{F}mm2$	2×2	···	Full	···
67	$mmm1'\mathrm{F}m'm2'$	2×2	P	Full	···
68	$mmm1'\mathrm{F}m'm'2$	2×2	P	Full	···
69	$mmm1'\mathrm{F}mm21'$	2	···	Full	···
70	$mmm1'\mathrm{F}mmm$	1×2	···	···	···
71	$mmm1'\mathrm{F}mmm'$	1×2	···	···	···
72	$mmm1'\mathrm{F}m'm'm$	1×2	Full	···	···
73	$mmm1'\mathrm{F}m'm'm'$	1×2	···	···	···
74	$41'\mathrm{F}1$	4×2	Full	Full	Full
75	$41'\mathrm{F}1'$	4	···	Full	Full
76	$41'\mathrm{F}2$	2×2	P	···	Full
77	$41'\mathrm{F}2'$	2×2	Full	···	Full
78	$41'\mathrm{F}21'$	2	···	···	Full
79	$41'\mathrm{F}4$	1×2	Full	···	···
80	$41'\mathrm{F}4'$	1×2	···	···	···
81	$\bar{4}1'\mathrm{F}1$	4×2	Full	Full	Full
82	$\bar{4}1'\mathrm{F}1'$	4	···	Full	Full
83	$\bar{4}1'\mathrm{F}2$	2×2	P	Full	Full
84	$\bar{4}1'\mathrm{F}2'$	2×2	Full	Full	Full
85	$\bar{4}1'\mathrm{F}21'$	2	···	Full	Full
86	$\bar{4}1'\mathrm{F}\bar{4}$	1×2	Full	···	···
87	$\bar{4}1'\mathrm{F}\bar{4}'$	1×2	···	···	···
88	$4/m1'\mathrm{F}1$	8×2	P	Full	P
89	$4/m1'\mathrm{F}1'$	8	···	Full	P
90	$4/m1'\mathrm{F}\bar{1}$	4×2	Full	···	Full
91	$4/m1'\mathrm{F}\bar{1}'$	4×2	···	···	Full
92	$4/m1'\mathrm{F}\bar{1}1'$	4	···	···	Full
93	$4/m1'\mathrm{F}2$	4×2	P	P	P
94	$4/m1'\mathrm{F}2'$	4×2	P	P	P
95	$4/m1'\mathrm{F}21'$	4	···	P	P
96	$4/m1'\mathrm{F}m$	4×2	P	Full	P
97	$4/m1'\mathrm{F}m'$	4×2	P	Full	P
98	$4/m1'\mathrm{F}m1'$	4	···	Full	P
99	$4/m1'\mathrm{F}2/m$	2×2	P	···	Full
100	$4/m1'\mathrm{F}2'/m$	2×2	···	···	Full
101	$4/m1'\mathrm{F}2/m'$	2×2	···	···	Full
102	$4/m1'\mathrm{F}2'/m'$	2×2	Full	···	Full
103	$4/m1'\mathrm{F}2/m1'$	2	···	···	Full
104	$4/m1'\mathrm{F}4$	2×2	P	Full	···
105	$4/m1'\mathrm{F}4'$	2×2	···	Full	···
106	$4/m1'\mathrm{F}41'$	2	···	Full	···
107	$4/m1'\mathrm{F}\bar{4}$	2×2	P	···	···
108	$4/m1'\mathrm{F}\bar{4}'$	2×2	···	···	···
109	$4/m1'\mathrm{F}\bar{4}1'$	2	···	···	···
110	$4/m1'\mathrm{F}4/m$	1×2	Full	···	···
111	$4/m1'\mathrm{F}4/m'$	1×2	···	···	···
112	$4/m1'\mathrm{F}4'/m$	1×2	···	···	···

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-magnetic	Ferro-electric	Ferro-elastic
113	4/m1'F4'/m'	1×2	···	···	···
114	4221'F1	8×2	Full	Full	Full
115	4221'F1'	8	···	Full	Full
116	4221'F2(p)	4×2	P	P	Full
117	4221'F2(s)	4×2	P	Full	Full
118	4221'F2'(p)	4×2	Full	P	Full
119	4221'F2'(s)	4×2	Full	Full	Full
120	4221'F21'(p)	4	···	P	Full
121	4221'F21'(s)	4	···	Full	Full
122	4221'F222	2×2	···	···	Full
123	4221'F2'2'2(p)	2×2	P	···	Full
124	4221'F2'2'2(s)	2×2	Full	···	Full
125	4221'F2221'	2	···	···	Full
126	4221'F4	2×2	P	Full	···
127	4221'F4'	2×2	···	Full	···
128	4221'F41'	2	···	Full	···
129	4221'F422	1×2	···	···	···
130	4221'F42'2'	1×2	Full	···	···
131	4221'F4'2'2	1×2	···	···	···
132	4mm1'F1	8×2	Full	Full	Full
133	4mm1'F1'	8	···	Full	Full
134	4mm1'F2	4×2	P	···	Full
135	4mm1'F2'	4×2	Full	···	Full
136	4mm1'F21'	4	···	···	Full
137	4mm1'Fm	4×2	P	Full	Full
138	4mm1'Fm'	4×2	Full	Full	Full
139	4mm1'Fm1'	4	···	Full	Full
140	4mm1'Fmm2	2×2	···	···	Full
141	4mm1'Fm'm2'	2×2	Full	···	Full
142	4mm1'Fm'm'2	2×2	P	···	Full
143	4mm1'Fmm21'	2	···	···	Full
144	4mm1'F4	2×2	P	···	···
145	4mm1'F4'	2×2	···	···	···
146	4mm1'F41'	2	···	···	···
147	4mm1'F4mm	1×2	···	···	···
148	4mm1'F4m'm'	1×2	Full	···	···
149	4mm1'F4'm'm	1×2	···	···	···
150	$\bar{4}$2m1'F1	8×2	Full	Full	Full
151	$\bar{4}$2m1'F1'	8	···	Full	Full
152	$\bar{4}$2m1'F2(p)	4×2	P	P	Full
153	$\bar{4}$2m1'F2(s)	4×2	P	Full	Full
154	$\bar{4}$2m1'F2'(p)	4×2	Full	P	Full
155	$\bar{4}$2m1'F2'(s)	4×2	Full	Full	Full
156	$\bar{4}$2m1'F21'(p)	4	···	P	Full
157	$\bar{4}$2m1'F21'(s)	4	···	Full	Full
158	$\bar{4}$2m1'Fm	4×2	P	Full	Full
159	$\bar{4}$2m1'Fm'	4×2	Full	Full	Full
160	$\bar{4}$2m1'Fm1'	4	···	Full	Full
161	$\bar{4}$2m1'F222	2×2	···	···	Full
162	$\bar{4}$2m1'F2'2'2(p)	2×2	P	···	Full
163	$\bar{4}$2m1'F2'2'2(s)	2×2	Full	···	Full
164	$\bar{4}$2m1'F2221'	2	···	···	Full
165	$\bar{4}$2m1'Fmm2	2×2	···	Full	Full
166	$\bar{4}$2m1'Fm'm2'	2×2	Full	Full	Full
167	$\bar{4}$2m1'Fm'm'2	2×2	P	Full	Full
168	$\bar{4}$2m1'Fmm21'	2	···	Full	Full
169	$\bar{4}$2m1'F$\bar{4}$	2×2	P	···	···
170	$\bar{4}$2m1'F$\bar{4}$'	2×2	···	···	···

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-magnetic	Ferro-electric	Ferro-elastic
171	$\bar{4}$2m1'F$\bar{4}$1'	2	···	···	···
172	$\bar{4}$2m1'F$\bar{4}$2m	1×2	···	···	···
173	$\bar{4}$2m1'F$\bar{4}$2'm'	1×2	Full	···	···
174	$\bar{4}$2m1'F$\bar{4}$'2'm	1×2	···	···	···
175	$\bar{4}$2m1'F$\bar{4}$'2m'	1×2	···	···	···
176	4/mmm1'F1	16×2	P	Full	P
177	4/mmm1'F1'	16	···	Full	P
178	4/mmm1'F$\bar{1}$	8×2	Full	···	Full
179	4/mmm1'F$\bar{1}$'	8×2	···	···	Full
180	4/mmm1'F$\bar{1}$1'	8	···	···	Full
181	4/mmm1'F2(p)	8×2	P	P	P
182	4/mmm1'F2(s)	8×2	P	P	P
183	4/mmm1'F2'(p)	8×2	P	P	P
184	4/mmm1'F2'(s)	8×2	P	P	P
185	4/mmm1'F21'(p)	8	···	P	P
186	4/mmm1'F21'(s)	8	···	P	P
187	4/mmm1'Fm(p)	8×2	P	Full	P
188	4/mmm1'Fm(s)	8×2	P	Full	P
189	4/mmm1'Fm'(p)	8×2	P	Full	P
190	4/mmm1'Fm'(s)	8×2	P	Full	P
191	4/mmm1'Fm1'(p)	8	···	Full	P
192	4/mmm1'Fm1'(s)	8	···	Full	P
193	4/mmm1'F2/m(p)	4×2	P	···	Full
194	4/mmm1'F2/m(s)	4×2	P	···	Full
195	4/mmm1'F2'/m(p)	4×2	···	···	Full
196	4/mmm1'F2'/m(s)	4×2	···	···	Full
197	4/mmm1'F2/m'(p)	4×2	···	···	Full
198	4/mmm1'F2/m'(s)	4×2	···	···	Full
199	4/mmm1'F2'/m'(p)	4×2	Full	···	Full
200	4/mmm1'F2'/m'(s)	4×2	Full	···	Full
201	4/mmm1'F2/m1'(p)	4	···	···	Full
202	4/mmm1'F2/m1'(s)	4	···	···	Full
203	4/mmm1'F222	4×2	···	···	P
204	4/mmm1'F2'2'2(p)	4×2	P	···	P
205	4/mmm1'F2'2'2(s)	4×2	P	···	P
206	4/mmm1'F2221'	4	···	···	P
207	4/mmm1'Fmm2(p)	4×2	···	P	P
208	4/mmm1'Fmm2(s)	4×2	···	Full	P
209	4/mmm1'Fm'm2'(ss)	4×2	P	P	P
210	4/mmm1'Fm'm2'(ps)	4×2	P	Full	P
211	4/mmm1'Fm'm2'(sp)	4×2	P	Full	P
212	4/mmm1'Fm'm'2(p)	4×2	P	P	P
213	4/mmm1'Fm'm'2(s)	4×2	P	Full	P
214	4/mmm1'Fmm21'(p)	4	···	P	P
215	4/mmm1'Fmm21'(s)	4	···	Full	P
216	4/mmm1'Fmmm	2×2	···	···	Full
217	4/mmm1'Fmmm'(p)	2×2	···	···	Full
218	4/mmm1'Fmmm'(s)	2×2	···	···	Full
219	4/mmm1'Fm'm'm(p)	2×2	P	···	Full
220	4/mmm1'Fm'm'm(s)	2×2	Full	···	Full
221	4/mmm1'Fm'm'm'	2×2	···	···	Full
222	4/mmm1'Fmmm1'	2	···	···	Full
223	4/mmm1'F4	4×2	P	P	···
224	4/mmm1'F4'	4×2	···	P	···
225	4/mmm1'F41'	4	···	P	···
226	4/mmm1'F$\bar{4}$	4×2	P	···	···
227	4/mmm1'F$\bar{4}$'	4×2	···	···	···
228	4/mmm1'F$\bar{4}$1'	4	···	···	···
229	4/mmm1'F4/m	2×2	P	···	···

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-magnetic	Ferro-electric	Ferro-elastic
230	$4/mmm1'\mathrm{F}4/m'$	2×2	···	···	···
231	$4/mmm1'\mathrm{F}4'/m$	2×2	···	···	···
232	$4/mmm1'\mathrm{F}4'/m'$	2×2	···	···	···
233	$4/mmm1'\mathrm{F}4/m1'$	2	···	···	···
234	$4/mmm1'\mathrm{F}422$	2×2	···	···	···
235	$4/mmm1'\mathrm{F}42'2'$	2×2	P	···	···
236	$4/mmm1'\mathrm{F}4'2'2$	2×2	···	···	···
237	$4/mmm1'\mathrm{F}4221'$	2	···	···	···
238	$4/mmm1'\mathrm{F}4mm$	2×2	···	Full	···
239	$4/mmm1'\mathrm{F}4m'm'$	2×2	P	Full	···
240	$4/mmm1'\mathrm{F}4'm'm$	2×2	···	Full	···
241	$4/mmm1'\mathrm{F}4mm1'$	2	···	Full	···
242	$4/mmm1'\mathrm{F}\bar{4}2m$	2×2	···	···	···
243	$4/mmm1'\mathrm{F}\bar{4}2'm'$	2×2	P	···	···
244	$4/mmm1'\mathrm{F}\bar{4}'2'm$	2×2	···	···	···
245	$4/mmm1'\mathrm{F}\bar{4}'2m'$	2×2	···	···	···
246	$4/mmm1'\mathrm{F}\bar{4}2m1'$	2	···	···	···
247	$4/mmm1'\mathrm{F}4/mmm$	1×2	···	···	···
248	$4/mmm1'\mathrm{F}4/mm'm'$	1×2	Full	···	···
249	$4/mmm1'\mathrm{F}4/m'mm$	1×2	···	···	···
250	$4/mmm1'\mathrm{F}4/m'm'm'$	1×2	···	···	···
251	$4/mmm1'\mathrm{F}4'/mm'm$	1×2	···	···	···
252	$4/mmm1'\mathrm{F}4'/m'm'm$	1×2	···	···	···
253	$31'\mathrm{F}1$	3×2	Full	Full	Full
254	$31'\mathrm{F}1'$	3	···	Full	Full
255	$31'\mathrm{F}3$	1×2	Full	···	···
256	$\bar{3}1'\mathrm{F}1$	6×2	P	Full	P
257	$\bar{3}1'\mathrm{F}1'$	6	···	Full	P
258	$\bar{3}1'\mathrm{F}\bar{1}$	3×2	Full	···	Full
259	$\bar{3}1'\mathrm{F}\bar{1}'$	3×2	···	···	Full
260	$\bar{3}1'\mathrm{F}\bar{1}1'$	3	···	···	Full
261	$\bar{3}1'\mathrm{F}3$	2×2	P	Full	···
262	$\bar{3}1'\mathrm{F}31'$	2	···	Full	···
263	$\bar{3}1'\mathrm{F}\bar{3}$	1×2	Full	···	···
264	$\bar{3}1'\mathrm{F}\bar{3}'$	1×2	···	···	···
265	$321'\mathrm{F}1$	6×2	Full	Full	Full
266	$321'\mathrm{F}1'$	6	···	Full	Full
267	$321'\mathrm{F}2$	3×2	Full	Full	Full
268	$321'\mathrm{F}2'$	3×2	Full	Full	Full
269	$321'\mathrm{F}21'$	3	···	Full	Full
270	$321'\mathrm{F}3$	2×2	P	Full	···
271	$321'\mathrm{F}31'$	2	···	Full	···
272	$321'\mathrm{F}32$	1×2	···	···	···
273	$321'\mathrm{F}32'$	1×2	Full	···	···
274	$3m1'\mathrm{F}1$	6×2	Full	Full	Full
275	$3m1'\mathrm{F}1'$	6	···	Full	Full
276	$3m1'\mathrm{F}m$	3×2	Full	Full	Full
277	$3m1'\mathrm{F}m'$	3×2	Full	Full	Full
278	$3m1'\mathrm{F}m1'$	3	···	Full	Full
279	$3m1'\mathrm{F}3$	2×2	P	···	···
280	$3m1'\mathrm{F}31'$	2	···	···	···
281	$3m1'\mathrm{F}3m$	1×2	···	···	···
282	$3m1'\mathrm{F}3m'$	1×2	Full	···	···
283	$\bar{3}m1'\mathrm{F}1$	12×2	P	Full	P
284	$\bar{3}m1'\mathrm{F}1'$	12	···	Full	P
285	$\bar{3}m1'\mathrm{F}\bar{1}$	6×2	Full	···	Full
286	$\bar{3}m1'\mathrm{F}\bar{1}'$	6×2	···	···	Full
287	$\bar{3}m1'\mathrm{F}\bar{1}1'$	6	···	···	Full
288	$\bar{3}m1'\mathrm{F}2$	6×2	P	Full	P
289	$\bar{3}m1'\mathrm{F}2'$	6×2	P	Full	P
290	$\bar{3}m1'\mathrm{F}21'$	6	···	Full	P
291	$\bar{3}m1'\mathrm{F}m$	6×2	P	Full	P
292	$\bar{3}m1'\mathrm{F}m'$	6×2	P	Full	P
293	$\bar{3}m1'\mathrm{F}m1'$	6	···	Full	P
294	$\bar{3}m1'\mathrm{F}2/m$	3×2	Full	···	Full
295	$\bar{3}m1'\mathrm{F}2'/m$	3×2	···	···	Full
296	$\bar{3}m1'\mathrm{F}2/m'$	3×2	···	···	Full
297	$\bar{3}m1'\mathrm{F}2'/m'$	3×2	Full	···	Full
298	$\bar{3}m1'\mathrm{F}2/m1'$	3	···	···	Full
299	$\bar{3}m1'\mathrm{F}3$	4×2	P	P	···
300	$\bar{3}m1'\mathrm{F}31'$	4	···	P	···
301	$\bar{3}m1'\mathrm{F}\bar{3}$	2×2	P	···	···
302	$\bar{3}m1'\mathrm{F}\bar{3}'$	2×2	···	···	···
303	$\bar{3}m1'\mathrm{F}\bar{3}1'$	2	···	···	···
304	$\bar{3}m1'\mathrm{F}32$	2×2	···	···	···
305	$\bar{3}m1'\mathrm{F}32'$	2×2	P	···	···
306	$\bar{3}m1'\mathrm{F}321'$	2	···	···	···
307	$\bar{3}m1'\mathrm{F}3m$	2×2	···	Full	···
308	$\bar{3}m1'\mathrm{F}3m'$	2×2	P	Full	···
309	$\bar{3}m1'\mathrm{F}3m1'$	2	···	Full	···
310	$\bar{3}m1'\mathrm{F}\bar{3}m$	1×2	···	···	···
311	$\bar{3}m1'\mathrm{F}\bar{3}m'$	1×2	Full	···	···
312	$\bar{3}m1'\mathrm{F}\bar{3}'m$	1×2	···	···	···
313	$\bar{3}m1'\mathrm{F}\bar{3}'m'$	1×2	···	···	···
314	$61'\mathrm{F}1$	6×2	Full	Full	Full
315	$61'\mathrm{F}1'$	6	···	Full	Full
316	$61'\mathrm{F}2$	3×2	P	···	Full
317	$61'\mathrm{F}2'$	3×2	Full	···	Full
318	$61'\mathrm{F}21'$	3	···	···	Full
319	$61'\mathrm{F}3$	2×2	P	···	···
320	$61'\mathrm{F}31'$	2	···	···	···
321	$61'\mathrm{F}6$	1×2	Full	···	···
322	$61'\mathrm{F}6'$	1×2	···	···	···
323	$\bar{6}1'\mathrm{F}1$	6×2	Full	Full	Full
324	$\bar{6}1'\mathrm{F}1'$	6	···	Full	Full
325	$\bar{6}1'\mathrm{F}m$	3×2	P	Full	Full
326	$\bar{6}1'\mathrm{F}m'$	3×2	Full	Full	Full
327	$\bar{6}1'\mathrm{F}m1'$	3	···	Full	Full
328	$\bar{6}1'\mathrm{F}3$	2×2	P	Full	···
329	$\bar{6}1'\mathrm{F}31'$	2	···	Full	···
330	$\bar{6}1'\mathrm{F}\bar{6}$	1×2	Full	···	···
331	$\bar{6}1'\mathrm{F}\bar{6}'$	1×2	···	···	···
332	$6/m1'\mathrm{F}1$	12×2	P	Full	P
333	$6/m1'\mathrm{F}1'$	12	···	Full	P
334	$6/m1'\mathrm{F}\bar{1}$	6×2	Full	···	Full
335	$6/m1'\mathrm{F}\bar{1}'$	6×2	···	···	Full
336	$6/m1'\mathrm{F}\bar{1}1'$	6	···	···	Full
337	$6/m1'\mathrm{F}2$	6×2	P	P	P
338	$6/m1'\mathrm{F}2'$	6×2	P	P	P
339	$6/m1'\mathrm{F}21'$	6	···	P	P
340	$6/m1'\mathrm{F}m$	6×2	P	Full	P
341	$6/m1'\mathrm{F}m'$	6×2	P	Full	P
342	$6/m1'\mathrm{F}m1'$	6	···	Full	P
343	$6/m1'\mathrm{F}2/m$	3×2	P	···	Full
344	$6/m1'\mathrm{F}2'/m$	3×2	···	···	Full

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-magnetic	Ferro-electric	Ferro-elastic
345	6/m1′F2/m′	3×2	⋯	⋯	Full
346	6/m1′F2′/m′	3×2	Full	⋯	Full
347	6/m1′F2/m1′	3	⋯	⋯	Full
348	6/m1′F3	4×2	P	P	⋯
349	6/m1′F31′	4	⋯	P	⋯
350	6/m1′F$\bar{3}$	2×2	P	⋯	⋯
351	6/m1′F$\bar{3}$′	2×2	⋯	⋯	⋯
352	6/m1′F$\bar{3}$1′	2	⋯	⋯	⋯
353	6/m1′F6	2×2	P	Full	⋯
354	6/m1′F6′	2×2	⋯	Full	⋯
355	6/m1′F61′	2	⋯	Full	⋯
356	6/m1′F$\bar{6}$	2×2	P	⋯	⋯
357	6/m1′F$\bar{6}$′	2×2	⋯	⋯	⋯
358	6/m1′F$\bar{6}$1′	2	⋯	⋯	⋯
359	6/m1′F6/m	1×2	Full	⋯	⋯
360	6/m1′F6/m′	1×2	⋯	⋯	⋯
361	6/m1′F6′/m	1×2	⋯	⋯	⋯
362	6/m1′F6′/m′	1×2	⋯	⋯	⋯
363	6221′F1	12×2	Full	Full	Full
364	6221′F1′	12	⋯	Full	Full
365	6221′F2(p)	6×2	P	P	Full
366	6221′F2(s)	6×2	P	Full	Full
367	6221′F2′(p)	6×2	Full	P	Full
368	6221′F2′(s)	6×2	Full	Full	Full
369	6221′F21′(p)	6	⋯	P	Full
370	6221′F21′(s)	6	⋯	Full	Full
371	6221′F222	3×2	⋯	⋯	Full
372	6221′F2′2′2(p)	3×2	P	⋯	Full
373	6221′F2′2′2(s)	3×2	Full	⋯	Full
374	6221′F2221′	3	⋯	⋯	Full
375	6221′F3	4×2	P	P	⋯
376	6221′F31′	4	⋯	P	⋯
377	6221′F32	2×2	⋯	⋯	⋯
378	6221′F32′	2×2	P	⋯	⋯
379	6221′F321′	2	⋯	⋯	⋯
380	6221′F6	2×2	P	Full	⋯
381	6221′F6′	2×2	⋯	Full	⋯
382	6221′F61′	2	⋯	Full	⋯
383	6221′F622	1×2	⋯	⋯	⋯
384	6221′F62′2′	1×2	Full	⋯	⋯
385	6221′F6′2′2	1×2	⋯	⋯	⋯
386	6mm1′F1	12×2	Full	Full	Full
387	6mm1′F1′	12	⋯	Full	Full
388	6mm1′F2	6×2	P	⋯	Full
389	6mm1′F2′	6×2	Full	⋯	Full
390	6mm1′F21′	6	⋯	⋯	Full
391	6mm1′Fm	6×2	P	Full	Full
392	6mm1′Fm′	6×2	Full	Full	Full
393	6mm1′Fm1′	6	⋯	Full	Full
394	6mm1′Fmm2	3×2	⋯	⋯	Full
395	6mm1′Fm′m2′	3×2	Full	⋯	Full
396	6mm1′Fm′m′2	3×2	P	⋯	Full
397	6mm1′Fmm21′	3	⋯	⋯	Full
398	6mm1′F3	4×2	P	⋯	⋯
399	6mm1′F31′	4	⋯	⋯	⋯
400	6mm1′F3m	2×2	⋯	⋯	⋯
401	6mm1′F3m′	2×2	P	⋯	⋯
402	6mm1′F3m1′	2	⋯	⋯	⋯
403	6mm1′F6	2×2	P	⋯	⋯
404	6mm1′F6′	2×2	⋯	⋯	⋯
405	6mm1′F61′	2	⋯	⋯	⋯
406	6mm1′F6mm	1×2	⋯	⋯	⋯
407	6mm1′F6m′m′	1×2	Full	⋯	⋯
408	6mm1′F6′m′m	1×2	⋯	⋯	⋯
409	$\bar{6}$m21′F1	12×2	Full	Full	Full
410	$\bar{6}$m21′F1′	12	⋯	Full	Full
411	$\bar{6}$m21′F2	6×2	P	P	Full
412	$\bar{6}$m21′F2′	6×2	Full	P	Full
413	$\bar{6}$m21′F21′	6	⋯	P	Full
414	$\bar{6}$m21′Fm(p)	6×2	P	Full	Full
415	$\bar{6}$m21′Fm(s)	6×2	P	Full	Full
416	$\bar{6}$m21′Fm′(p)	6×2	Full	Full	Full
417	$\bar{6}$m21′Fm′(s)	6×2	Full	Full	Full
418	$\bar{6}$m21′Fm1′(p)	6	⋯	Full	Full
419	$\bar{6}$m21′Fm1′(s)	6	⋯	Full	Full
420	$\bar{6}$m21′Fmm2	3×2	⋯	Full	Full
421	$\bar{6}$m21′Fm′m2′(ps)	3×2	Full	Full	Full
422	$\bar{6}$m21′Fm′m2′(sp)	3×2	P	Full	Full
423	$\bar{6}$m21′Fm′m′2	3×2	Full	Full	Full
424	$\bar{6}$m21′Fmm21′	3	⋯	Full	Full
425	$\bar{6}$m21′F3	4×2	P	P	⋯
426	$\bar{6}$m21′F31′	4	⋯	P	⋯
427	$\bar{6}$m21′F32	2×2	⋯	⋯	⋯
428	$\bar{6}$m21′F32′	2×2	P	⋯	⋯
429	$\bar{6}$m21′F321′	2	⋯	⋯	⋯
430	$\bar{6}$m21′F3m	2×2	⋯	Full	⋯
431	$\bar{6}$m21′F3m′	2×2	P	Full	⋯
432	$\bar{6}$m21′F3m1′	2	⋯	Full	⋯
433	$\bar{6}$m21′F$\bar{6}$	2×2	P	⋯	⋯
434	$\bar{6}$m21′F$\bar{6}$′	2×2	⋯	⋯	⋯
435	$\bar{6}$m21′F$\bar{6}$1′	2	⋯	⋯	⋯
436	$\bar{6}$m21′F$\bar{6}$m2	1×2	⋯	⋯	⋯
437	$\bar{6}$m21′F$\bar{6}$m′2′	1×2	Full	⋯	⋯
438	$\bar{6}$m21′F$\bar{6}$′m2′	1×2	⋯	⋯	⋯
439	$\bar{6}$m21′F$\bar{6}$′m′2	1×2	⋯	⋯	⋯
440	6/mmm1′F1	24×2	P	Full	P
441	6/mmm1′F1′	24	⋯	Full	P
442	6/mmm1′F$\bar{1}$	12×2	Full	⋯	Full
443	6/mmm1′F$\bar{1}$′	12×2	⋯	⋯	Full
444	6/mmm1′F$\bar{1}$1′	12	⋯	⋯	Full
445	6/mmm1′F2(p)	12×2	P	P	P
446	6/mmm1′F2(s)	12×2	P	P	P
447	6/mmm1′F2′(p)	12×2	P	P	P
448	6/mmm1′F2′(s)	12×2	P	P	P
449	6/mmm1′F21′(p)	12	⋯	P	P
450	6/mmm1′F21′(s)	12	⋯	P	P
451	6/mmm1′Fm(p)	12×2	P	Full	P
452	6/mmm1′Fm(s)	12×2	P	Full	P
453	6/mmm1′Fm′(p)	12×2	P	Full	P
454	6/mmm1′Fm′(s)	12×2	P	Full	P
455	6/mmm1′Fm1′(p)	12	⋯	Full	P
456	6/mmm1′Fm1′(s)	12	⋯	Full	P
457	6/mmm1′F2/m(p)	6×2	P	⋯	Full
458	6/mmm1′F2/m(s)	6×2	P	⋯	Full
459	6/mmm1′F2′/m(p)	6×2	⋯	⋯	Full
460	6/mmm1′F2′/m(s)	6×2	⋯	⋯	Full

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-mag-netic	Ferro-elec-tric	Ferro-elas-tic
461	6/mmm1′F2/m′(p)	6×2	···	···	Full
462	6/mmm1′F2/m′(s)	6×2	···	···	Full
463	6/mmm1′F2′/m′(p)	6×2	Full	···	Full
464	6/mmm1′F2′/m′(s)	6×2	Full	···	Full
465	6/mmm1′F2/m1′(p)	6	···	···	Full
466	6/mmm1′F2/m1′(s)	6	···	···	Full
467	6/mmm1′F222	6×2	···	···	P
468	6/mmm1′F2′2′2(p)	6×2	P	···	P
469	6/mmm1′F2′2′2(s)	6×2	P	···	P
470	6/mmm1′F2221′	6	···	···	P
471	6/mmm1′Fmm2(p)	6×2	···	P	P
472	6/mmm1′Fmm2(s)	6×2	···	Full	P
473	6/mmm1′Fm′m2′(ss)	6×2	P	P	P
474	6/mmm1′Fm′m2′(ps)	6×2	P	Full	P
475	6/mmm1′Fm′m2′(sp)	6×2	P	Full	P
476	6/mmm1′Fm′m′2(p)	6×2	P	P	P
477	6/mmm1′Fm′m′2(s)	6×2	P	Full	P
478	6/mmm1′Fmm21′(p)	6	···	P	P
479	6/mmm1′Fmm21′(s)	6	···	Full	P
480	6/mmm1′Fmmm	3×2	···	···	Full
481	6/mmm1′Fmmm′(p)	3×2	···	···	Full
482	6/mmm1′Fmmm′(s)	3×2	···	···	Full
483	6/mmm1′Fm′m′m(p)	3×2	P	···	Full
484	6/mmm1′Fm′m′m(s)	3×2	Full	···	Full
485	6/mmm1′Fm′m′m′	3×2	···	···	Full
486	6/mmm1′Fmmm1′	3	···	···	Full
487	6/mmm1′F3	8×2	P	P	···
488	6/mmm1′F31′	8	···	P	···
489	6/mmm1′F$\bar{3}$	4×2	P	···	···
490	6/mmm1′F$\bar{3}$′	4×2	···	···	···
491	6/mmm1′F$\bar{3}$1′	4	···	···	···
492	6/mmm1′F32	4×2	···	···	···
493	6/mmm1′F32′	4×2	P	···	···
494	6/mmm1′F321′	4	···	···	···
495	6/mmm1′F3m	4×2	···	P	···
496	6/mmm1′F3m′	4×2	P	P	···
497	6/mmm1′F3m1′	4	···	P	···
498	6/mmm1′F$\bar{3}$m	2×2	···	···	···
499	6/mmm1′F$\bar{3}$m′	2×2	P	···	···
500	6/mmm1′F$\bar{3}$′m	2×2	···	···	···
501	6/mmm1′F$\bar{3}$′m′	2×2	···	···	···
502	6/mmm1′F$\bar{3}$m1′	2	···	···	···
503	6/mmm1′F6	4×2	P	P	···
504	6/mmm1′F6′	4×2	···	P	···
505	6/mmm1′F61′	4	···	P	···
506	6/mmm1′F$\bar{6}$	4×2	P	···	···
507	6/mmm1′F$\bar{6}$′	4×2	···	···	···
508	6/mmm1′F$\bar{6}$1′	4	···	···	···
509	6/mmm1′F6/m	2×2	P	···	···
510	6/mmm1′F6/m′	2×2	···	···	···
511	6/mmm1′F6′/m	2×2	···	···	···
512	6/mmm1′F6′/m′	2×2	···	···	···
513	6/mmm1′F6/m1′	2	···	···	···
514	6/mmm1′F622	2×2	···	···	···
515	6/mmm1′F62′2′	2×2	P	···	···
516	6/mmm1′F6′2′2	2×2	···	···	···
517	6/mmm1′F6221′	2	···	···	···
518	6/mmm1′F6mm	2×2	···	Full	···
519	6/mmm1′F6m′m′	2×2	P	Full	···
520	6/mmm1′F6′m′m	2×2	···	Full	···
521	6/mmm1′F6mm1′	2	···	Full	···
522	6/mmm1′F$\bar{6}$m2	2×2	···	···	···
523	6/mmm1′F$\bar{6}$m′2′	2×2	P	···	···
524	6/mmm1′F$\bar{6}$′m2′	2×2	···	···	···
525	6/mmm1′F$\bar{6}$′m′2	2×2	···	···	···
526	6/mmm1′F$\bar{6}$m21′	2	···	···	···
527	6/mmm1′F6/mmm	1×2	···	···	···
528	6/mmm1′F6/mm′m′	1×2	Full	···	···
529	6/mmm1′F6/m′mm	1×2	···	···	···
530	6/mmm1′F6/m′m′m′	1×2	···	···	···
531	6/mmm1′F6′/mm′m	1×2	···	···	···
532	6/mmm1′F6′/m′m′m	1×2	···	···	···
533	231′F1	12×2	Full	Full	Full
534	231′F1′	12	···	Full	Full
535	231′F2	6×2	P	Full	Full
536	231′F2′	6×2	Full	Full	Full
537	231′F21′	6	···	Full	Full
538	231′F222	3×2	···	···	Full
539	231′F2′2′2	3×2	Full	···	Full
540	231′F2221′	3	···	···	Full
541	231′F3	4×2	Full	Full	Full
542	231′F31′	4	···	Full	Full
543	231′F23	1×2	···	···	···
544	m31′F1	24×2	P	Full	P
545	m31′F1′	24	···	Full	P
546	m31′F$\bar{1}$	12×2	Full	···	Full
547	m31′F$\bar{1}$′	12×2	···	···	Full
548	m31′F$\bar{1}$1′	12	···	···	Full
549	m31′F2	12×2	P	P	P
550	m31′F2′	12×2	P	P	P
551	m31′F21′	12	···	P	P
552	m31′Fm	12×2	P	Full	P
553	m31′Fm′	12×2	P	Full	P
554	m31′Fm1′	12	···	Full	P
555	m31′F2/m	6×2	P	···	Full
556	m31′F2′/m	6×2	···	···	Full
557	m31′F2/m′	6×2	···	···	Full
558	m31′F2′/m′	6×2	Full	···	Full
559	m31′F2/m1′	6	···	···	Full
560	m31′F222	6×2	···	···	P
561	m31′F2′2′2	6×2	P	···	P
562	m31′F2221′	6	···	···	P
563	m31′Fmm2	6×2	···	Full	P
564	m31′Fm′m2′	6×2	P	Full	P
565	m31′Fm′m′2	6×2	P	Full	P
566	m31′Fmm21′	6	···	Full	P
567	m31′Fmmm	3×2	···	···	Full
568	m31′Fmmm′	3×2	···	···	Full
569	m31′Fm′m′m	3×2	Full	···	Full
570	m31′Fm′m′m′	3×2	···	···	Full
571	m31′Fmmm1′	3	···	···	Full
572	m31′F3	8×2	P	Full	P
573	m31′F31′	8	···	Full	P
574	m31′F$\bar{3}$	4×2	Full	···	Full
575	m31′F$\bar{3}$′	4×2	···	···	Full
576	m31′F$\bar{3}$1′	4	···	···	Full
577	m31′F23	2×2	···	···	···

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-magnetic	Ferro-electric	Ferro-elastic
578	$m31'$F$231'$	2	···	···	···
579	$m31'$F$m3$	1×2	···	···	···
580	$m31'$F$m'3$	1×2	···	···	···
581	$4321'$F1	24×2	Full	Full	Full
582	$4321'$F$1'$	24	···	Full	Full
583	$4321'$F$2(p)$	12×2	P	P	Full
584	$4321'$F$2(s)$	12×2	P	Full	Full
585	$4321'$F$2'(p)$	12×2	Full	P	Full
586	$4321'$F$2'(s)$	12×2	Full	Full	Full
587	$4321'$F$21'(p)$	12	···	P	Full
588	$4321'$F$21'(s)$	12	···	Full	Full
589	$4321'$F$222(pp)$	6×2	···	···	Full
590	$4321'$F$222(ss)$	6×2	···	···	Full
591	$4321'$F$2'2'2(pp)$	6×2	P	···	Full
592	$4321'$F$2'2'2(ss)$	6×2	P	···	Full
593	$4321'$F$2'2'2(ps)$	6×2	Full	···	Full
594	$4321'$F$2221'(pp)$	6	···	···	Full
595	$4321'$F$2221'(ss)$	6	···	···	Full
596	$4321'$F4	6×2	P	Full	P
597	$4321'$F$4'$	6×2	···	Full	P
598	$4321'$F$41'$	6	···	Full	P
599	$4321'$F422	3×2	···	···	Full
600	$4321'$F$42'2'$	3×2	Full	···	Full
601	$4321'$F$4'2'2$	3×2	···	···	Full
602	$4321'$F$4221'$	3	···	···	Full
603	$4321'$F3	8×2	P	Full	P
604	$4321'$F$31'$	8	···	Full	P
605	$4321'$F32	4×2	···	···	Full
606	$4321'$F$32'$	4×2	Full	···	Full
607	$4321'$F$321'$	4	···	···	Full
608	$4321'$F23	2×2	···	···	···
609	$4321'$F$231'$	2	···	···	···
610	$4321'$F432	1×2	···	···	···
611	$4321'$F$4'32'$	1×2	···	···	···
612	$\bar{4}3m1'$F1	24×2	Full	Full	Full
613	$\bar{4}3m1'$F$1'$	24	···	Full	Full
614	$\bar{4}3m1'$F2	12×2	P	P	Full
615	$\bar{4}3m1'$F$2'$	12×2	Full	P	Full
616	$\bar{4}3m1'$F$21'$	12	···	P	Full
617	$\bar{4}3m1'$Fm	12×2	P	Full	Full
618	$\bar{4}3m1'$Fm'	12×2	Full	Full	Full
619	$\bar{4}3m1'$F$m1'$	12	···	Full	Full
620	$\bar{4}3m1'$F222	6×2	···	···	Full
621	$\bar{4}3m1'$F$2'2'2$	6×2	P	···	Full
622	$\bar{4}3m1'$F$2221'$	6	···	···	Full
623	$\bar{4}3m1'$F$mm2$	6×2	···	Full	Full
624	$\bar{4}3m1'$F$m'm2'$	6×2	Full	Full	Full
625	$\bar{4}3m1'$F$m'm'2$	6×2	P	Full	Full
626	$\bar{4}3m1'$F$mm21'$	6	···	Full	Full
627	$\bar{4}3m1'$F$\bar{4}$	6×2	P	···	P
628	$\bar{4}3m1'$F$\bar{4}'$	6×2	···	···	P
629	$\bar{4}3m1'$F$\bar{4}1'$	6	···	···	P
630	$\bar{4}3m1'$F$\bar{4}2m$	3×2	···	···	Full
631	$\bar{4}3m1'$F$\bar{4}2'm'$	3×2	Full	···	Full
632	$\bar{4}3m1'$F$\bar{4}'2'm$	3×2	···	···	Full
633	$\bar{4}3m1'$F$\bar{4}'2m'$	3×2	···	···	Full
634	$\bar{4}3m1'$F$\bar{4}2m1'$	3	···	···	Full
635	$\bar{4}3m1'$F3	8×2	P	P	P
636	$\bar{4}3m1'$F$31'$	8	···	P	P
637	$\bar{4}3m1'$F$3m$	4×2	···	Full	Full
638	$\bar{4}3m1'$F$3m'$	4×2	Full	Full	Full
639	$\bar{4}3m1'$F$3m1'$	4	···	Full	Full
640	$\bar{4}3m1'$F23	2×2	···	···	···
641	$\bar{4}3m1'$F$231'$	2	···	···	···
642	$\bar{4}3m1'$F$\bar{4}3m$	1×2	···	···	···
643	$\bar{4}3m1'$F$\bar{4}'3m'$	1×2	···	···	···
644	$m3m1'$F1	48×2	P	Full	P
645	$m3m1'$F$1'$	48	···	Full	P
646	$m3m1'$F$\bar{1}$	24×2	Full	···	Full
647	$m3m1'$F$\bar{1}'$	24×2	···	···	Full
648	$m3m1'$F$\bar{1}1'$	24	···	···	Full
649	$m3m1'$F$2(p)$	24×2	P	P	P
650	$m3m1'$F$2(s)$	24×2	P	P	P
651	$m3m1'$F$2'(p)$	24×2	P	P	P
652	$m3m1'$F$2'(s)$	24×2	P	P	P
653	$m3m1'$F$21'(p)$	24	···	P	P
654	$m3m1'$F$21'(s)$	24	···	P	P
655	$m3m1'$F$m(p)$	24×2	P	Full	P
656	$m3m1'$F$m(s)$	24×2	P	Full	P
657	$m3m1'$F$m'(p)$	24×2	P	Full	P
658	$m3m1'$F$m'(s)$	24×2	P	Full	P
659	$m3m1'$F$m1'(p)$	24	···	Full	P
660	$m3m1'$F$m1'(s)$	24	···	Full	P
661	$m3m1'$F$2/m(p)$	12×2	P	···	Full
662	$m3m1'$F$2/m(s)$	12×2	P	···	Full
663	$m3m1'$F$2'/m(p)$	12×2	···	···	Full
664	$m3m1'$F$2'/m(s)$	12×2	···	···	Full
665	$m3m1'$F$2/m'(p)$	12×2	···	···	Full
666	$m3m1'$F$2/m'(s)$	12×2	···	···	Full
667	$m3m1'$F$2'/m'(p)$	12×2	Full	···	Full
668	$m3m1'$F$2'/m'(s)$	12×2	Full	···	Full
669	$m3m1'$F$2/m1'(p)$	12	···	···	Full
670	$m3m1'$F$2/m1'(s)$	12	···	···	Full
671	$m3m1'$F$222(pp)$	12×2	···	···	P
672	$m3m1'$F$222(ss)$	12×2	···	···	P
673	$m3m1'$F$2'2'2(pp)$	12×2	P	···	P
674	$m3m1'$F$2'2'2(ss)$	12×2	P	···	P
675	$m3m1'$F$2'2'2(ps)$	12×2	P	···	P
676	$m3m1'$F$2221'(pp)$	12	···	···	P
677	$m3m1'$F$2221'(ss)$	12	···	···	P
678	$m3m1'$F$mm2(pp)$	12×2	···	P	P
679	$m3m1'$F$mm2(ss)$	12×2	···	P	P
680	$m3m1'$F$mm2(ps)$	12×2	···	Full	P
681	$m3m1'$F$m'm2'(pp)$	12×2	P	P	P
682	$m3m1'$F$m'm2'(ss)$	12×2	P	P	P
683	$m3m1'$F$m'm2'(ps)$	12×2	P	Full	P
684	$m3m1'$F$m'm2'(sp)$	12×2	P	Full	P
685	$m3m1'$F$m'm'2(pp)$	12×2	P	P	P
686	$m3m1'$F$m'm'2(ss)$	12×2	P	P	P
687	$m3m1'$F$m'm'2(ps)$	12×2	P	Full	P
688	$m3m1'$F$mm21'(pp)$	12	···	P	P
689	$m3m1'$F$mm21'(ss)$	12	···	P	P
690	$m3m1'$F$mm21'(ps)$	12	···	Full	P
691	$m3m1'$F$mmm(pp)$	6×2	···	···	Full
692	$m3m1'$F$mmm(ss)$	6×2	···	···	Full
693	$m3m1'$F$mmm'(pp)$	6×2	···	···	Full
694	$m3m1'$F$mmm'(ss)$	6×2	···	···	Full

TABLE I. (*Continued*)

No.	Species	Number of states	Ferro-magnetic	Ferro-electric	Ferro-elastic
695	m3m1′Fmmm′(ps)	6×2	···	···	Full
696	m3m1′Fm′m′m(pp)	6×2	P	···	Full
697	m3m1′Fm′m′m(ss)	6×2	P	···	Full
698	m3m1′Fm′m′m(ps)	6×2	Full	···	Full
699	m3m1′Fm′m′m′(pp)	6×2	···	···	Full
700	m3m1′Fm′m′m′(ss)	6×2	···	···	Full
701	m3m1′Fmmm1′(pp)	6	···	···	Full
702	m3m1′Fmmm1′(ss)	6	···	···	Full
703	m3m1′F4	12×2	P	P	P
704	m3m1′F4′	12×2	···	P	P
705	m3m1′F41′	12	···	P	P
706	m3m1′F$\bar{4}$	12×2	P	···	P
707	m3m1′F$\bar{4}$′	12×2	···	···	P
708	m3m1′F$\bar{4}$1′	12	···	···	P
709	m3m1′F4/m	6×2	P	···	P
710	m3m1′F4/m′	6×2	···	···	P
711	m3m1′F4′/m	6×2	···	···	P
712	m3m1′F4′/m′	6×2	···	···	P
713	m3m1′F4/m1′	6	···	···	P
714	m3m1′F422	6×2	···	···	P
715	m3m1′F42′2′	6×2	P	···	P
716	m3m1′F4′2′2(ps)	6×2	···	···	P
717	m3m1′F4′2′2(sp)	6×2	···	···	P
718	m3m1′F4221′	6	···	···	P
719	m3m1′F4mm	6×2	···	Full	P
720	m3m1′F4m′m′	6×2	P	Full	P
721	m3m1′F4′m′m(ps)	6×2	···	Full	P
722	m3m1′F4′m′m(sp)	6×2	···	Full	P
723	m3m1′F4mm1′	6	···	Full	P
724	m3m1′F$\bar{4}$2m(ps)	6×2	···	···	P
725	m3m1′F$\bar{4}$2m(sp)	6×2	···	···	P
726	m3m1′F$\bar{4}$2′m′(ps)	6×2	P	···	P
727	m3m1′F$\bar{4}$2′m′(sp)	6×2	P	···	P
728	m3m1′F$\bar{4}$′2′m(ps)	6×2	···	···	P
729	m3m1′F$\bar{4}$′2′m(sp)	6×2	···	···	P
730	m3m1′F$\bar{4}$′2m′(ps)	6×2	···	···	P
731	m3m1′F$\bar{4}$′2m′(sp)	6×2	···	···	P
732	m3m1′F$\bar{4}$2m1′(ps)	6	···	···	P
733	m3m1′F$\bar{4}$2m1′(sp)	6	···	···	P
734	m3m1′F4/mmm	3×2	···	···	Full
735	m3m1′F4/mm′m′	3×2	Full	···	Full
736	m3m1′F4/m′mm	3×2	···	···	Full
737	m3m1′F4/m′m′m′	3×2	···	···	Full
738	m3m1′F4′/mm′m(ps)	3×2	···	···	Full
739	m3m1′F4′/mm′m(sp)	3×2	···	···	Full
740	m3m1′F4′/m′m′m(ps)	3×2	···	···	Full
741	m3m1′F4′/m′m′m(sp)	3×2	···	···	Full
742	m3m1′F4/mmm1′	3	···	···	Full
743	m3m1′F3	16×2	P	P	P
744	m3m1′F31′	16	···	P	P
745	m3m1′F$\bar{3}$	8×2	P	···	P
746	m3m1′F$\bar{3}$′	8×2	···	···	P
747	m3m1′F$\bar{3}$1′	8	···	···	P
748	m3m1′F32	8×2	···	···	P
749	m3m1′F32′	8×2	P	···	P
750	m3m1′F321′	8	···	···	P
751	m3m1′F3m	8×2	···	Full	P
752	m3m1′F3m′	8×2	P	Full	P
753	m3m1′F3m1′	8	···	Full	P
754	m3m1′F$\bar{3}$m	4×2	···	···	Full
755	m3m1′F$\bar{3}$m′	4×2	Full	···	Full
756	m3m1′F$\bar{3}$′m	4×2	···	···	Full
757	m3m1′F$\bar{3}$′m′	4×2	···	···	Full
758	m3m1′F$\bar{3}$m1′	4	···	···	Full
759	m3m1′F23	4×2	···	···	···
760	m3m1′F231′	4	···	···	···
761	m3m1′Fm3	2×2	···	···	···
762	m3m1′Fm′3	2×2	···	···	···
763	m3m1′Fm31′	2	···	···	···
764	m3m1′F432	2×2	···	···	···
765	m3m1′F4′32′	2×2	···	···	···
766	m3m1′F4321′	2	···	···	···
767	m3m1′F$\bar{4}$3m	2×2	···	···	···
768	m3m1′F$\bar{4}$′3m′	2×2	···	···	···
769	m3m1′F$\bar{4}$3m1′	2	···	···	···
770	m3m1′Fm3m	1×2	···	···	···
771	m3m1′Fm3m′	1×2	···	···	···
772	m3m1′Fm′3m	1×2	···	···	···
773	m3m1′Fm′3m′	1×2	···	···	···

In No. 445, the diad axis of FG is along the hexad axis of PG; No. 446, the diad axis of FG is along a diad axis perpendicular to the hexad axis of PG; Nos. 447, 449, similar to No. 445; Nos. 448, 450, similar to No. 446; No. 451, the mirror plane of FG is along the mirror plane perpendicular to the hexad axis of PG; No. 452, the mirror plane of FG is along a mirror plane parallel to the hexad axis of PG; Nos. 453, 455, similar to No. 451; Nos. 454, 456, similar to No. 452; No. 457, the diad axis of FG is along the hexad axis of PG or, in other words, the mirror plane of FG is along the mirror plane perpendicular to the hexad axis of PG; No. 458, the diad axis of FG is along a diad axis perpendicular to the hexad axis of PG or, in other words, the mirror plane of FG is along a mirror plane parallel to the hexad axis of PG; Nos. 459, 461, 463, 465, similar to No. 457; Nos. 460, 462, 464, 466, similar to No. 458; No. 468, the pure diad axis of FG is along the hexad axis of PG; No. 469, the pure diad axis of FG is along a diad axis perpendicular to the hexad axis of PG; No. 471, the diad axis of FG is along the hexad axis of PG; No. 472, the diad axis of FG is along a diad axis perpendicular to the hexad axis of PG; No. 473, the m' plane of FG is along a mirror plane parallel to the hexad axis of PG, and the pure mirror plane of FG is also along a mirror plane parallel to the hexad axis of PG (the $2'$ axis of FG is along the hexad axis of PG); No. 474, the m' plane of FG is along the mirror plane perpendicular to the hexad axis of PG, and the pure mirror plane of FG is along a mirror plane parallel to the hexad axis of PG (the 2′ axis of FG is along a diad axis perpendicular to the hexad axis of PG); No. 475, the m' plane of FG is along a mirror plane parallel to the hexad axis of PG, and the pure mirror plane of FG is along the mirror plane perpendicular to the hexad axis of PG (the 2′ axis of FG is along a diad axis perpendicular to the hexad axis of PG); Nos. 476, 478, similar to No. 471; Nos. 477, 479, similar to No. 472; No. 481, the m' plane of FG is along the mirror plane perpendicular to the hexad axis of PG; No. 482, the m' plane of FG is along a mirror plane parallel to the hexad axis of PG; No. 483, the pure mirror plane of FG is along the mirror plane perpendicular to the hexad axis of PG; No. 484, the pure mirror plane of FG is along a mirror plane parallel to the hexad axis of PG.

In No. 583, the diad axis of FG is along a tetrad axis of PG; No. 584, the diad axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG; Nos. 585, 587, similar to No. 583; Nos. 586, 588, similar to No. 584; No. 589, two of the diad axes of FG are along tetrad axes of PG (the remaining diad axis of FG is also along a tetrad axis of PG); No. 590, two of the diad axes of FG are along diad axes, making an angle of 45° to tetrad axes, of PG (the remaining diad axis of FG is along a tetrad axis of PG); No. 591, both the 2′ axes of FG are along tetrad axes of PG (the pure diad axis of FG is also along a tetrad axis of PG); No. 592, both the 2′ axes of FG are along diad axes, making an angle of 45° to tetrad axes, of PG (the pure diad axis of FG is along a tetrad axis of PG); No. 593, one of the 2′ axes of FG is along a tetrad axis of PG, and the other 2′ axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG (the pure diad axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG); No. 594, similar to No. 589; No. 595, similar to No. 590.

In No. 649, the diad axis of FG is along a tetrad axis of PG; No. 650, the diad axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG; Nos. 651, 653, similar to No. 649; Nos. 652, 654, similar to No. 650; No. 655, the mirror plane of FG is along a mirror plane perpendicular to a tetrad axis of PG; No. 656, the mirror plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG; Nos. 657, 659, similar to No. 655; Nos. 658, 660, similar to No. 656; No. 661, the diad axis of FG is along a tetrad axis of PG or, in other words, the mirror plane of FG is along a mirror plane perpendicular to a tetrad axis of PG; No. 662, the diad axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG or, in other words, the mirror plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG; Nos. 663, 665, 667, 669, similar to No. 661; Nos. 664, 666, 668, 670, similar to No. 662; No. 671, two of the diad axes of FG are along tetrad axes of PG (the remaining diad axis of FG is also along a tetrad axis of PG); No. 672, two of the diad axes of FG are along diad axes, making an angle of 45° to tetrad axes, of PG (the remaining diad axis of FG is along a tetrad axis of PG); No. 673, both the 2′ axes of FG are along tetrad axes of PG (the pure diad axis of FG is also along a tetrad axis of PG); No. 674, both the 2′ axes of FG are along diad axes, making an angle of 45° to tetrad axes, of PG (the pure diad axis of FG is along a tetrad axis of PG); No. 675, one of the 2′ axes of FG is along a tetrad axis of PG, and the other 2′ axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG (the pure diad axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG); No. 676, similar to No. 671; No. 677, similar to No. 672; No. 678, both the mirror planes of FG are along mirror planes perpendicular to tetrad axes of PG (the diad axis of FG is along a tetrad axis of PG); No. 679, both the mirror planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG (the diad axis of FG is along a tetrad axis of PG); No. 680, one of the mirror planes of FG is along a mirror plane perpendicular to a tetrad axis of PG, and the other mirror plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG (the diad axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG); No. 681, the m' plane of FG is along a mirror plane perpendicular to a tetrad axis of PG, and the pure mirror plane of FG is also along a mirror plane perpendicular to a tetrad axis of PG (the 2′ axis of FG is along a tetrad axis of PG); No. 682, the m' plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG, and the pure mirror plane of FG is also along a mirror plane, making an angle of 45° to a tetrad axis, of PG (the 2′ axis of FG is along a tetrad axis of PG); No. 683, the m' plane of FG is along a mirror plane perpendicular to a tetrad axis of PG, and the pure mirror plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG (the 2′ axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG); No. 684, the m' plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG, and the pure mirror plane of FG is along a mirror plane perpendicular to a tetrad axis of PG (the 2′ axis of FG is along a diad axis, making an angle of 45° to a tetrad axis, of PG); Nos. 685, 688, similar to No. 678; Nos. 686, 689, similar to No. 679; Nos. 687, 690, similar to No. 680; No. 691, two of the mirror planes of FG are along mirror planes perpendicular to tetrad axes of PG (the remaining mirror plane of FG is also along a mirror plane perpendicular to a tetrad axis of PG); No. 692, two of the mirror planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG (the remaining mirror plane of FG is along a mirror plane perpendicular to a tetrad axis of PG); No. 693, both the pure mirror planes of FG are along mirror planes perpendicular to tetrad axes of PG (the m' plane of FG is also along a mirror plane perpendicular to a tetrad axis of PG); No. 694, both the pure mirror planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG (the m' plane of FG is along a mirror plane perpendicular to a tetrad axis of PG); No. 695, one of the pure mirror planes of FG is along a mirror plane perpendicular to a tetrad axis of PG, and the other pure mirror plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG (the m' plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG); No. 696, both the m' planes of FG are along mirror planes perpendicular to tetrad axes of PG (the pure mirror plane of FG is also along a mirror plane perpendicular to a tetrad axis of PG); No. 697, both the m' planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG (the pure mirror plane of FG is along a mirror plane perpendicular to a tetrad axis of PG); No. 698, one of the m' planes of FG is along a mirror plane perpendicular to a tetrad axis of PG, and the other m' plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG (the pure mirror plane of FG is along a mirror plane, making an angle of 45° to a tetrad axis, of PG); Nos. 699, 701, similar to No. 691; Nos. 700, 702, similar to No. 692; No. 716, the $2'$ axes of FG are along tetrad axes of PG, and the pure diad axes (except the one parallel to the $4'$ axis) of FG are along diad axes, making an angle of 45° to tetrad axes, of PG; No. 717, the $2'$ axes of FG are along diad axes, making an angle of 45° to tetrad axes, of PG, and the pure diad axes (except the one parallel to the $4'$ axis) of FG are along tetrad axes of PG; No. 721, the m' planes of FG are along mirror planes perpendicular to tetrad axes of PG, and the pure mirror planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG; No. 722, the m' planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG, and the pure mirror planes of FG are along mirror planes perpendicular to tetrad axes of PG; No. 724, the diad axes (except the one parallel to the $\bar{4}$ axis) of FG are along tetrad axes of PG, and the mirror planes of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG; No. 725, the diad axes (except the one paralled to the $\bar{4}$ axis) of FG are along diad axes, making an angle of 45° to tetrad axes, of PG, and the mirror planes of FG are along mirror planes perpendicular to tetrad axes of PG; Nos. 726, 728, 730, 732, similar to No. 724; Nos. 727, 729, 731, 733, similar to No. 725; No. 738, the m' planes parallel to the $4'$ axis of FG are along mirror planes perpendicular to tetrad axes of PG, and the pure mirror planes parallel to the $4'$ axis of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG; No. 739, the m' planes parallel to the $4'$ axis of FG are along mirror planes, making an angle of 45° to tetrad axes, of PG, and the pure mirror planes parallel to the $4'$ axis of FG are along mirror planes perpendicular to tetrad axes of PG; No. 740, similar to No. 738; No. 741, similar to No. 739.

5. RELATIONSHIPS OF EACH SPECIES TO FERROMAGNETISM, FERROELECTRICITY, AND FERROELASTICITY

We explain the way of determining whether a particular species is full, partially, or not ferromagnetic, with species $\bar{4}2m1'$F$mm2$, $\bar{4}2m1'$F$m'm2'$, and $\bar{4}2m1'$F$m'm'2$ as examples. First, since the point group $mm2$ cannot possess a spontaneous magnetization vector,[4] ferroics of the species $\bar{4}2m1'$F$mm2$ must all be nonferromagnetic.

The point group $m'm2'$ can possess a nonzero spontaneous magnetization vector in the direction perpendicular to the pure mirror plane.[4] We set a system of rectangular coordinate axes x, y, z with the z axis along the $\bar{4}$ axis in the prototypic point group $\bar{4}2m1'$; the x axis may be either perpendicular to one of the mirror planes or parallel to one of the diad axes in the prototypic point group:

$$\begin{aligned} &z \parallel \text{the tetragonal axis (prot.)},\\ &x \perp \text{a mirror plane (prot.)},\\ &y \perp \text{a mirror plane (prot.)}, \end{aligned} \qquad (1a)$$

or

$$\begin{aligned} &z \parallel \text{the tetragonal axis (prot.)},\\ &x \parallel \text{a diad axis (prot.)},\\ &y \parallel \text{a diad axis (prot.)}. \end{aligned} \qquad (1b)$$

We choose the system (1a) and designate an orientation state with spontaneous magnetization vector toward $+x$ as S_1. Then $\{1, 1', \bar{4}, \bar{4}'\}$ is found to be a set of representative F operations on S_1. Here $1'$ stands for time inversion (not the time-inversion group); $\bar{4}$ stands for the combination of the 90° space rotation about the z axis and space inversion; $\bar{4}'$ stands for the combination of $1'$ and $\bar{4}$. Since the operations $1'$, $\bar{4}$, and $\bar{4}'$ turn the spontaneous magnetization vector from $+x$ to $-x$, $+y$, and $-y$, respectively, the states S_1, $1'S_1$, $\bar{4}S_1$, and $\bar{4}'S_1$ are all different with respect to direction of spontaneous magnetization vector. Therefore it is concluded that ferroics of the species $\bar{4}2m1'\mathrm{F}m'm2'$ are, in general, full ferromagnetic.

The point group $m'm'2$ can possess a nonzero spontaneous magnetization vector along its diad axis.[4] We set the system of rectangular coordinate axes (1a) and designate an orientation state with spontaneous magnetization vector toward $+z$ as S_1. Then, $\{1, 1', \bar{4}, \bar{4}'\}$ is found to be a set of representative F operations on S_1. Since the operations $1'$, $\bar{4}$, and $\bar{4}'$ turn the spontaneous magnetization vector from $+z$ to $-z$, $+z$ itself, and $-z$, respectively, the state S_1 differs from the state $1'S_1$ but not from the state $\bar{4}S_1$ with respect to direction of spontaneous magnetization vector. Therefore it is concluded that ferroics of the species $\bar{4}2m1'\mathrm{F}m'm'2$ are, in general, partially ferromagnetic.

The relationships of a species GFH^*, where H^* is not time symmetric, to ferroelectricity and ferroelasticity are the same as those of the species GFH where $H = H^* \times 1'$. Thus it is recommended that the reader consult Ref. 1 for the way of finding such relationships. Here we only consider ferroelectricity in the species $4/mmm1'\mathrm{F}21'(s)$. The point group $21'$ can possess a nonzero spontaneous polarization vector along its diad axis. We set the system of rectangular coordinate axes (1). [When $4/mmm1'$ is the prototypic point group, the systems (1a) and (1b) are the same.] We designate an orientation state with spontaneous polarization vector toward $+x$ as S_1. The result from performance of space inversion upon S_1 must be a possible orientation state different from S_1. We designate this state as S_2. Since space inversion turns the spontaneous polarization vector from $+x$ to $-x$, the states S_1 and S_2 are opposite in spontaneous polarization vector. Denote the space reflection across the xy plane by m_z. The result from performance of m_z upon S_1 must be a possible orientation state different from S_1. We designate this state as S_3. Since m_z turns the spontaneous polarization vector being in the $+x$ direction to no other direction, S_3 is the same as S_1 in spontaneous polarization vector. S_3 is not time conjugate with S_1. After all, it is evident that ferroics of the species $4/mmm1'\mathrm{F}21'(s)$ are, in general, partially ferroelectric.

The last three columns of Table I shown the relationships of each species to ferromagnetism, ferroelectricity, and ferroelasticity. The letter P is an abbreviation of partial. There are, in all, 327 ferromagnetic species, 126 of which are full. There are, in all, 333 ferroelectric species, 243 of which are full. There are, in all, 513 ferroelastic species, 320 of which are full. Since spontaneous magnetization vector is invariant under space inversion, any species with a centrosymmetric prototypic point group and with a noncentrosymmetric ferroic point group cannot be full ferromagnetic. It is a matter of course that any species which has a non-time-symmetric ferroic point group and is two in number of orientation states is neither ferroelectric nor ferroelastic. The species which are full ferromagnetic and possess more than two orientation states are all full ferroelastic. This is well known. However, a partially ferromagnetic species (that always possesses more than two orientation states) may be full ferroelastic, partially ferroelastic, or nonferroelastic. In all simultaneously full ferromagnetic and full ferroelastic species, the coupling of spontaneous magnetization vector and spontaneous strain tensor is "complete" in the sense that a turn of spontaneous magnetization vector through an angle other than 180° due to a state shift by magnetic field is always accompanied by a change of spontaneous strain tensor and, conversely, a change of spontaneous strain tensor due to a state shift by mechanical stress is always accompanied by a turn of spontaneous magnetization vector through an angle other than 180°. In all partially ferromagnetic and full ferroelastic species and in all partially ferromagnetic and partially ferroelastic species, the coupling of spontaneous magnetization vector and spontaneous strain tensor is incomplete.

Consider, for example, a partially ferromagnetic and full ferroelastic species $4/mmm1'\mathrm{F}2/m(s)$. We set the system of rectangular coordinate axes (1). Spontaneous magnetization vector must be along the diad axis in the ferroic point group,[4] and this diad axis is along one of the diad axes perpendicular to the tetrad axis in the prototypic point group. We designate an orientation state with spontaneous magnetization vector toward $+x$ as S_{+1} and designate the results from performance of the 90°, 180°, and 270° space rotations about the z axis upon S_{+1} as S_{+2}, S_{+3}, and S_{+4}, respectively. S_{+1}, S_{+2}, S_{+3}, and S_{+4} must, obviously, be possible orientation states different from and not even time conjugate to one another. We denote the orientation states time conjugate to them by S_{-1}, S_{-2}, S_{-3}, and S_{-4}, respectively. These eight orientation states are all of the orientation states in the relevant species. We can easily find: The spontaneous strain tensor in S_{+1} and S_{-1} has the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & d \\ 0 & d & c \end{pmatrix},$$

where a, b, c, d are nonzero and different from one another; the spontaneous strain tensors are

$$\begin{pmatrix} b & 0 & -d \\ 0 & a & 0 \\ -d & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & -d \\ 0 & -d & c \end{pmatrix}, \text{ and } \begin{pmatrix} b & 0 & d \\ 0 & a & 0 \\ d & 0 & c \end{pmatrix} .$$

in S_{+2} and S_{-2}, S_{+3} and S_{-3}, and in S_{+4} and S_{-4}, respectively. This shows explicitly the difference in spontaneous strain tensor among the non-time-conjugate orientation states. On the other hand, we can easily find that spontaneous magnetization vector points to $\pm x$, $\pm y$, $\mp x$, and $\mp y$ in $S_{\pm1}$, $S_{\pm2}$, $S_{\pm3}$, and $S_{\pm4}$, respectively. Therefore, a change of spontaneous strain tensor due to the shift from $S_{\pm1}$ to $S_{\pm2}$ by mechanical stress is accompained by a turn of spontaneous magnetization vector through a right angle, while a change of spontaneous strain tensor due to the shift from $S_{\pm1}$ to $S_{\pm3}$ by mechinical stress is accompanied by no turn of spontaneous magnetization vector. By magnetic fields alone, without the aid of mechanical stresses, it is difficult to reduce a crystal comprising, e.g., S_{+1} domains and S_{-3} domains to a single domain.

In all simultaneously full ferromagnetic and full ferroelectric species, the coupling of spontaneous magnetization vector and spontaneous polarization vector is "complete" in the sense that a turn of spontaneous magnetization vector through an angle other than 180° due to a state shift by magnetic field is always accompanied by a turn of spontaneous polarization vector (through 180° or another angle) and, conversely, a turn of spontaneous polarization vector (through 180° or another angle) due to a state shift by electric field is always accompanied by a turn of spontaneous magnetization vector through an angle other than 180°. In all full ferromagnetic and partially ferroelectric species, in all partially ferromagnetic and full ferroelectric species, and in all partially ferromagnetic and partially ferroelectric species, the coupling of spontaneous magnetization vector and spontaneous polarization vector is incomplete.

Consider, for example, a full ferromagnetic and full ferroelectric species $\bar{4}2m1'Fm'm2'$. We set the system of rectangular coordinate axes (1a). We designate an orientation state with spontaneous magnetization vector toward $+x$ as S_{+1} and designate the result from performance of the 180° space rotation about the [110] axis upon S_{+1} as S_{+2}. S_{+2} must, obviously, be a possible orientation state different from and not even time conjugate to S_{+1}. We denote the orientation states time conjugate to S_{+1} and S_{+2} by S_{-1} and S_{-2}, respectively. These four orientation states are all of the orientation states in the relevant species. It is easily found that spontaneous magnetization vector points to $+x$, $-x$, $+y$, $-y$ in S_{+1}, S_{-1}, S_{+2}, S_{-2}, respectively, and that spontaneous polarization vector points to $+z$ in $S_{\pm1}$ and to $-z$ in $S_{\pm2}$ (or to $-z$ in $S_{\pm1}$ and to $+z$ in $S_{\pm2}$). Therefore a turn of spontaneous magnetization vector through a right angle due to the shift from $S_{\pm1}$ to $S_{\pm2}$ by magnetic field is accompanied by a reversal of spontaneous polarization vector and, conversely, a reversal of spontaneous polarization vector due to the shift from $S_{\pm1}$ to $S_{\pm2}$ by electric field is accompanied by a turn of spontaneous magnetization vector through a right angle.

Consider another full ferromagnetic and full ferroelectric species $321'F2$. While in the species $\bar{4}2m1'Fm'm2'$ the spontaneous magnetization vector and spontaneous polarization vector are perpendicular in any orientation state, in the species $321'F2$ both vectors are parallel or antiparallel. In this species, although spontaneous magnetization vector can be reversed, spontaneous polarization vector cannot be reversed but can only be turned through 120°. A turn of spontaneous polarization vector through 120° by electric field should always be accompanied by a turn of spontaneous magnetization vector through 60° or 120° and, conversely, a turn of spontaneous magnetization vector through 60° or 120° by magnetic field should always be accompanied by a turn of spontaneous polarization vector through 120°.

Consider a partially ferromagnetic and full ferroelectric species $4/mmm1'Fm'm2'(ps)$. We set the system of rectangular coordinate axes (1). We designate an orientation state with spontaneous polarization vector toward $+x$ as S_{+1}, and designate the results from performance of the 90°, 180°, and 270° space rotations about the z axis upon S_{+1} as S_{+2}, S_{+3}, and S_{+4}, respectively. S_{+1}, S_{+2}, S_{+3}, and S_{+4} must, obviously, be different and non-time-conjugate orientation states. We denote the orientation states time conjugate to them by S_{-1}, S_{-2}, S_{-3}, and S_{-4}, respectively. These eight orientation states are all of the orientation states in the relevant species. It is easily found that spontaneous polarization vector points to $+x$, $+y$, $-x$, $-y$ in $S_{\pm1}$, $S_{\pm2}$, $S_{\pm3}$, $S_{\pm4}$, respectively, and that spontaneous magnetization vector points to $\pm y$, $\mp x$, $\mp y$, $\pm x$ (or $\mp y$, $\pm x$, $\pm y$, $\mp x$) in $S_{\pm1}$, $S_{\pm2}$, $S_{\pm3}$, $S_{\pm4}$, respectively. Therefore a turn of spontaneous polarization vector through a right angle due to the shift from $S_{\pm1}$ to $S_{\pm2}$ by electric field is accompanied by a turn of spontaneous magnetization vector through a right angle, while a reversal of spontaneous polarization vector due to the shift from $S_{\pm1}$ to $S_{\pm3}$ by electric field is accompanied by no turn (no reversal) of spontaneous magnetization vector. By magnetic fields alone, without the aid of electric fields, it is difficult to reduce a crystal comprising, e.g., S_{+1} domains and S_{-3} domains to a single domain.

Consider another partially ferromagnetic and full ferroelectric species $4/mmm1'Fm'm2'(sp)$. In this species, spontaneous polarization vector and spontaneous magnetization vector are perpendicular; the former can be turned through 90° and 180°, while the latter can only be reversed. (The plane on which spontaneous polarization vector is turnable is normal to the ferromagnetic easy axis.) Both vectors do not couple; that is, a turn of spontaneous polarization vector through 90° or 180° by electric field is accompanied by no reversal of spontaneous magnetization vector and, conversely, a reversal of spontaneous magnetization vector by magnetic field is, of course, accompanied by no turn of spontaneous polarization vector.

Consider a full ferromagnetic and partially ferroelectric species $4221'F2'(p)$. We set the system of rectangular coordinate axes (1b). We designate the orientation states with spontaneous magnetization vector in the directions $[hk0]$, $[h\bar{k}0]$, $[\bar{k}h0]$, and $[kh0]$ as S_{+1}, S_{+2}, S_{+3}, and S_{+4}, respectively, and the orientation states time conjugate to them as S_{-1}, S_{-2}, S_{-3}, and S_{-4}, respectively; here h and k are nonzero and absolutely different. These eight orientation states are all of the orientation states in the relevant species. One F operation from S_{+1} to S_{+2} is the 180° space rotation about the x axis; one F operation from S_{+1} to S_{+3} is the 90° space rotation about the z axis; one F operation from S_{+1} to S_{+4} is the 180° space rotation about the [110] axis. Spontaneous polarization vector, therefore, points to $+z$, $-z$, $+z$, $-z$ (or $-z$, $+z$, $-z$, $+z$) in $S_{\pm1}$, $S_{\pm2}$, $S_{\pm3}$, $S_{\pm4}$, respectively. Thus it is evident that a turn of spontaneous magnetization vector through an angle other than either 90° or 180° due to the shift from $S_{\pm1}$ to $S_{\pm2}$ by magnetic field is accompanied by a reversal of spontaneous polarization vector, while a turn of spontaneous magnetization vector through a right angle due to the shift from $S_{\pm1}$ to $S_{\pm3}$ by magnetic field is accompanied by no reversal of spontaneous polarization vector. Electric field can change a multidomain state consisting of S_{+1} domains, S_{-1} domains, S_{+3} domains, and S_{-3} domains to a multidomain state consisting of S_{+2} domains, S_{-2} domains, S_{+4} domains, and S_{-4} domains, and can return the latter multidomain state to the former, but cannot reduce such a multidomain state to a state consisting of only S_{+1} domains and S_{-1} domains (for example).

Consider the species $41'F2'$. This can possess a nonzero spontaneous magnetization vector and a nonzero spontaneous polarization vector. These vectors are perpendicular; the spontaneous magnetization vector can be turned through 90° and 180°, but the spontaneous polarization vector cannot be turned through any angle – even through 180°. Thus, the species $41'F2'$ is ferromagnetic; however, it is not ferroelectric but only pyroelectric.

Consider a partially ferromagnetic, partially ferroelectric, and partially ferroelastic species $4/mmm1'Fm'm'2(p)$. We set the system of rectangular coordinate axes (1) and designate as S an orientation state in which spontaneous magnetization vector points to $+z$, spontaneous polarization vector points to $+z$, and the element x_{12} of spontaneous strain tensor is positive. It can be found that $\{1, \bar{1}, 4, \bar{4}, 1', \bar{1}', 4', \bar{4}'\}$ is a set of representative F operations on S. We indicate an orientation state by $(-++)$ if spontaneous magnetization vector points to $-z$, spontaneous polarization vector points to $+z$, and x_{12} is positive, by $(+-+)$ if spontaneous magnetization vector points to $+z$, spontaneous polarization vector points to $-z$, and x_{12} is positive, and so forth. Then the indices of all the orientation states are

$$S \cdots (+++), \quad 1'S \cdots (-++),$$
$$\bar{1}S \cdots (+-+), \quad \bar{1}'S \cdots (--+),$$
$$4S \cdots (++-), \quad 4'S \cdots (-+-),$$
$$\bar{4}S \cdots (+--), \quad \bar{4}'S \cdots (---).$$

Therefore, spontaneous magnetization vector, spontaneous polarization vector, and spontaneous strain tensor do not couple with one another; that is, a reversal of spontaneous magnetization vector by magnetic field or of spontaneous polarization vector by electric field or of spontaneous strain tensor by mechanical stress is accompanied by no reversal of either of the others. (If S is subjected, for example, to an electric field pointing to $-z$ in the absence of magnetic field and mechanical stress, S is expected to be changed to $\bar{1}S$ and not $\bar{4}S$, $\bar{1}'S$, nor $\bar{4}'S$.) By any one or any two of magnetic field, electric field, and mechanical stress, it is difficult or practically impossible to reduce a crystal comprising all the eight kinds of domains to a single domain, but by a combination of the three it is possible.

6. EXAMPLES OF REAL FERROIC CRYSTALS

The room-temperature phase of cobalt belongs to species $6/mmm1'F6/mm'm'$, being full ferromagnetic, nonferroelectric, and nonferroelastic. $LiH_3(SeO_3)_2$ belongs[5,7] to species $2/m1'Fm1'$, being nonferromagnetic, full ferroelectric, and nonferroelastic. The room-temperature phase of VO_2 belongs[8] to species $4/mmm1'F2/m1'(s)$, being nonferromagnetic, nonferroelectric, and full ferroelastic.

The room-temperature phase of $Gd_2(MoO_4)_3$ belongs[9] to species $\bar{4}2m1'Fmm21'$, being nonferromagnetic, full ferroelectric, and full ferroelastic. It has been experimentally confirmed that a reversal of spontaneous polarization vector by an electric field along the z axis is always accompanied by a reversal of the element x_{12} of spontaneous strain tensor and, conversely, a reversal of x_{12} by a uniaxial pressure against the (110) or $(1\bar{1}0)$ faces is always accompanied by a reversal of spontaneous polarization vector.

The room-temperature phase of $BaTiO_3$ belongs to species $m3m1'F4mm1'$, being nonferromagnetic, full ferroelectric, and partially ferroelastic. It is well known that a turn of spontaneous polarization vector through a right angle by electric field is accompanied by a change of spontaneous strain tensor, but a reversal of spontaneous polarization vector by electric field is accompanied by no change of spontaneous strain tensor.

The room-temperature phase of $NaBa_2Nb_5O_{15}$ belongs[10] to species $4/mmm1'Fmm21'(p)$, being nonferromagnetic, partially ferroelectric, and partially ferroelastic. It has been observed that spontaneous polarization vector and spontaneous strain tensor are turned separately from each other by electric field and by mechanical stress, respectively.

The room-temperature phase of iron belongs to species $m3m1'F4/mm'm'$, being full ferromagnetic, nonferroelectric, and full ferroelastic. It is well known that although a reversal of spontaneous magnetization vector by magnetic field is accompanied by no change of spontaneous strain tensor, a turn of spontaneous magnetization vector through a right angle by magnetic field is accompanied by a turn of spontaneous strain tensor and, conversely, a turn of spontaneous strain tensor by mechanical stress is accompanied by a turn of spontaneous magnetization vector through a right angle.

The room-temperature phase of α-Fe_2O_3 belongs[11] to species $\bar{3}m1'F2/m$, being full ferromagnetic, nonferroelectric, and full ferroelastic. This crystal is known as "weak" ferromagnetic.

The phase of Fe_3O_4 which occurs at temperatures below $-154\,^\circ C$ belongs[12] probably to species $m3m1'Fm'm'm(ss)$, being partially ferromagnetic, nonferroelectric, and full ferroelastic. It is expected that when spontaneous magnetization vector is along, e.g., the z axis, a reversal of the element x_{12} of spontaneous strain tensor by a shear stress X_{12} is accompanied by no turn of spontaneous magnetization vector.

The phase of $Ni_3B_7O_{13}I$ (a kind of boracite) which occurs at temperatures below $64\,^\circ K$ belongs[13] to species $\bar{4}3m1'Fm'm2'$, being full ferromagnetic, full ferroelectric, and full ferroelastic. It has been observed that a reversal of spontaneous polarization vector by electric field is always accompanied by a turn of spontaneous magnetization vector through a right angle (in the plane perpendicular to spontaneous polarization vector) and, conversely, a turn of spontaneous magnetization vector through a right angle (in the plane perpendicular to spontaneous polarization vector) by magnetic field is always accompanied by a reversal of spontaneous polarization vector.

The phase of $GdBr_3$ which occurs below $2\,^\circ K$ may[14] possibly belong to species $321'F2$ that is full ferromagnetic, full ferroelectric, and full ferroelastic. (Compare Sec. 4 where this species has been discussed.) This crystal has been observed to be weak ferromagnetic with spontaneous magnetization vector normal to the trigonal axis, but has not yet been examined with respect to ferroelectricity.

Some of the 773 species of ferroic crystals shown in Table I have orientation states which may almost impossibly be changed to one another by any combination of a magnetic field, an electric field, and a mechanical stress. There are two notions of ferroic: One, which is adopted in the present paper, lays emphasis on state shift and the other on phase transformation. According to the first notion, such species may not be appropriate to call ferroic.[15] However, if the second notion were adopted, such species would be as ferroic as other species. Even when the first notion is adopted, it is often more convenient not to exclude such species.

The present theory deals with all of the ferromagnetics, ferroelectrics, and ferroelastics in a unified way upon a common basis, and may be especially helpful for investigating crystals in which ferromagnetism, ferroelectricity, and ferroelasticity coexist and couple completely or incompletely with each other.

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Effect of Single-Ion Anisotropy on Two-Spin-Wave Bound State in a Heisenberg Ferromagnet*

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We consider the scattering of two spin waves in a uniaxial (easy axis) Heisenberg ferromagnet with single-ion anisotropy. The two-spin-deviation problem is solved exactly at zero temperature. We find (for $S>\frac{1}{2}$), in addition to the usual two-spin-wave bound states, a new "single-ion bound state," in which at the zone corner the two spin deviations are on the same site. When the magnitude of the anisotropy is comparable to the exchange interaction, the single-ion bound state becomes the dominant feature of the bound-state spectrum. For arbitrary spin there is a critical anisotropy strength above which the single-ion bound state exists throughout the Brillouin zone. We conclude that the presence of single-ion anisotropy enhances the possibility of experimental observation of the bound states.

I. INTRODUCTION

The Heisenberg model of ferromagnetism has been extensively studied.[1] The elementary excitations of this model are the spin waves, which consist of single spin deviations propagating through the lattice.[2] Considering only the Ising part

$$-J\sum_{\langle i,j\rangle} S_i^z S_j^z$$

of the Heisenberg Hamiltonian, one finds that the excitation energy of two *adjacent* spin deviations is lower by J than that of two nonadjacent ones, giving rise to an effective attractive interaction between spin waves. Although the transverse terms in the Heisenberg Hamiltonian tend to weaken this attraction, it has been shown by Wortis[3] and Hanus[4] that the attractive interaction results in the formation of bound states of two spin waves for a sufficiently large total wave vector $\vec{q}$. Physically, these "exchange bound states" correspond to two spin deviations close together in space and propagating through the lattice in a correlated fashion with to-